Sparse Estimation of Multipath Biases for Global Navigation Satellite Systems

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Outline

Introduction

State Space Model

Sparse Estimation

Bayesian Estimation

Mixture Models

Conclusions and Future Works
Introduction
GPS Applications

LBS: Location-Based Services
80% of Smartphones

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GPS Applications

LBS: Location-Based Services
80% of Smartphones

Global Navigation Satellite Systems

- GPS: USA, 1973
- GLONASS: URSS, 1976
- Compass-Beidou: China, 1983 (Beidou) 2007 (Compass)
- Galileo: EU, 1999
- QZSS: Japan, 2002
- IRNSS: India, 2006
GNSS Satellites
GNSS Satellites

∼ 30 satellites/constellation
GNSS Satellites

~ 30 satellites/constellation

~ 20000 km altitude
Principle: trilateration
Principle: trilateration
Principle: trilateration
Signal propagation (1)
Signal propagation (1)
Signal propagation (1)

Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)
Signal propagation (1)

Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)

Ionosphere: electrons, ~ 50-1000 km
Signal propagation (1)

Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)

Ionosphere: electrons, \(\sim 50-1000 \text{ km}\)
Troposphere: gaz, \(\sim 12 \text{ km}\)
Signal propagation (2)
Signal propagation (2)
Signal propagation (2)
Signal propagation (2)
Signal propagation (2)
GNSS receiver

Antenna
GNSS receiver

Antenna

Electronics
GNSS receiver

Antenna

Electronics

Acquisition
GNSS receiver

Antenna

Electronics

Acquisition

DLL

FLL

PLL
GNSS receiver

Antenna

Electronics

Acquisition

DLL

FLL

PLL

Tracking
Introduction

State Space Model

Sparse Estimation

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Mixture Models

Conclusions

GNSS receiver

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Tracking

For a single satellite
GNSS receiver

Antenna

Electronics

Acquisition

DLL

FLL

PLL

Navigation

Tracking

For a single satellite
GNSS receiver

Introduction
State Space Model
Sparse Estimation
Bayesian Estimation
Mixture Models
Conclusions

Antenna -> Electronics -> Acquisition -> Navigation

Acquisition -> DLL -> Navigation

Acquisition -> FLL -> Navigation

Acquisition -> PLL -> Navigation

Position

Tracking

For a single satellite
GNSS receiver

Antenna

Electronics

Acquisition

 DLL

 FLL

 PLL

Raw measurements

Tracking

For a single satellite
State Space Model
Navigation problem

Measurements for satellite $i$ at time $k$

\[
\rho_{i,k} = \sqrt{\|r_k - r_{i,k}\|_2^2 + b_k + \varepsilon_{i,k}}
\]

\[
\dot{\rho}_{i,k} = (v_k - v_{i,k})^T u_{i,k} + \dot{b}_k + e_{i,k}
\]

$r_k$: receiver’s position

$r_{i,k}$: satellite’s position

$v_k$: receiver’s velocity

$v_{i,k}$: satellite’s velocity

$b_k$: receiver’s clock bias

$\dot{b}_k$: receiver’s clock drift

$\varepsilon_{i,k}$: pseudorange error

$e_{i,k}$: pseudospeed error

State vector:

\[
\xi_k = \{r_k, v_k, b_k, \dot{b}_k\} \in \mathbb{R}^8
\]
Navigation problem

Measurements for satellite $i$ at time $k$

$$\rho_{i,k} = \left\| r_k - r_{i,k} \right\|^2 + b_k + \varepsilon_{i,k}$$

$$\dot{\rho}_{i,k} = (v_k - v_{i,k})^T u_{i,k} + \dot{b}_k + e_{i,k}$$

$r_k$: receiver’s position   \hspace{1cm} r_{i,k}$: satellite’s position
$v_k$: receiver’s velocity   \hspace{1cm} v_{i,k}$: satellite’s velocity
$b_k$: receiver’s clock bias   \hspace{1cm} b_k$: receiver’s clock drift
$\varepsilon_{i,k}$: pseudorange error
$e_{i,k}$: pseudospeed error

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---

Navigation problem

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\]

$r_k$: receiver’s position
$v_k$: receiver’s velocity
$m_{i,k}$: orbit of $i$-th satellite
\(b_k\): receiver’s clock bias
\(\dot{b}_k\): receiver’s clock drift
\(\epsilon_{i,k}\): pseudorange error
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---

Navigation problem

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State vector:

$$\xi_k = \{r_k, v_k, b_k, \dot{b}_k\} \in \mathbb{R}^8$$

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GNSS Error Budget\textsuperscript{3}

\begin{itemize}
\item Satellite clock bias
\item Receiver clock bias
\item Receiver noise
\item Relativity
\item Ionosphere
\item Troposphere
\item Multipath
\item Ephemeris or Precise orbits
\item Explicit Klobuchar, bi-frequency or SBAS Niell
\item Estimated Statistical model
\end{itemize}

GNSS Error Budget\(^3\)

Satellite clock bias and position up to \(\sim 100\) km

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GNSS Error Budget

Satellite clock bias and position up to \( \sim 100 \text{ km} \)

Relativity \( \sim 10 \text{ m} \)

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GNSS Error Budget

- Satellite clock bias and position: up to ∼100 km
- Relativity: ∼10 m
- Ionosphere: 2-30 m

---

GNSS Error Budget

Satellite clock bias and position up to \( \sim 100 \) km

Relativity \( \sim 10 \) m

Ionosphere 2-30 m

Troposphere 2-30 m

---

GNSS Error Budget

- Satellite clock bias and position: up to ∼100 km
- Relativity: ∼10 m
- Ionosphere: 2-30 m
- Troposphere: 2-30 m
- Multipath: up to 450 m

---

GNSS Error Budget\(^3\)

- Satellite clock bias and position: up to \(\sim 100\) km
- Relativity: \(\sim 10\) m
- Ionosphere: 2-30 m
- Troposphere: 2-30 m
- Multipath: up to 450 m
- Receiver clock bias: \(\sim 300\) km

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GNSS Error Budget

- Satellite clock bias and position: up to ~100 km
- Relativity: ~10 m
- Ionosphere: 2-30 m
- Troposphere: 2-30 m
- Multipath: up to 450 m
- Receiver clock bias: ~300 km
- Receiver noise: ~1 m

---

GNSS Error Budget

Satellite clock bias and position
- Ephemeris or Precise orbits
  - up to ∼ 100 km
- Explicit

Relativity
- ∼ 10 m

Ionosphere
- 2-30 m
  - Klobuchar, bi-frequency or SBAS

Troposphere
- 2-30 m
  - Niell or SBAS

Multipath
- up to 450 m

Receiver clock bias
- ∼ 300 km

Receiver noise
- ∼ 1 m

Estimated

Statistical model

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Multipath Mitigation

Multipath Mitigation

Multipath Mitigation

Multipath Mitigation

Multipath Mitigation

Multipath Mitigation

GNSS signals
Code waveform

Multipath Mitigation

Antenna

Electronics

Acquisition

Raw measurements

DLL

Discriminator

Correlators

GNSS signals

Code waveform

Antenna

Geometry or spatial processing

Multipath Mitigation

GNSS signals
Code waveform
Antenna
Geometry or spatial processing
Digital signal
ML methods, DPE

Multipath Mitigation

**GNSS signals**
- Code waveform

**Antenna**
- Geometry or spatial processing

**Digital signal**
- ML methods, DPE

**Correlators**
- Narrow correlator, Multi-correlator

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Multipath Mitigation

GNSS signals
Code waveform

Antenna
Geometry or spatial processing

Digital signal
ML methods, DPE

Correlators
Narrow correlator, Multi-correlator

Raw measurements
Long term observation
Statistical methods

Multipath Mitigation

**GNSS signals**
Code waveform

**Antenna**
Geometry or spatial processing

**Digital signal**
ML methods, DPE

**Correlators**
Narrow correlator, Multi-correlator

**Raw measurements**
Long term observation

**Statistical methods**

---

System equations

**Measurements** $z_k \in \mathbb{R}^{2s_k}$

Hypothesis: models for everything except multipath and noise\(^5,6,7\)

\[ z_k = h_k(\xi_k) + m_k + n_k \quad \text{with} \quad h_k \, \text{known and nonlinear} \]

\[ m_k \, \text{unknown} \]

\[ n_k \sim \mathcal{N}(n_k; 0, R_k) \]

Extended Kalman Filter (EKF)
Filter considering a state propagation model (standard: $m_k = 0$)
Fault Detection and Exclusion (FDE)
Remove faulty satellites based on hypothesis tests on the residuals

---


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$z_k = h_k(\xi_k) + m_k + n_k$ \quad with \quad $h_k$ known and nonlinear \quad $m_k$ unknown \quad $n_k \sim \mathcal{N}(n_k; 0, R_k)$

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Standard EKF
Standard EKF

Legend

- Reference
- Standard EKF

Google Earth

500 m
Fault Detection and Exclusion (FDE)
Fault Detection and Exclusion (FDE)
First idea

\[ z_k = h_k(\xi_k) + m_k + n_k \]

Assumption: 6 satellites \( \equiv \) 12 measurements
\( \rightarrow \) maybe 4 measurements suffer from MP

\[ m_k = \begin{pmatrix} \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \end{pmatrix} \quad \rightarrow \quad m_k = \begin{pmatrix} 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \end{pmatrix} \]

\( \rightarrow \) Sparse estimation to estimate MP biases on raw measurements
First idea

\[ z_k = h_k(\xi_k) + m_k + n_k \]

**Assumption:** 6 satellites \(\equiv\) 12 measurements
\(\rightarrow\) maybe 4 measurements suffer from MP

\[ m_k = \begin{pmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \end{pmatrix} \]

\(\rightarrow\) Sparse estimation to estimate MP biases on raw measurements
First idea

\[ z_k = h_k(\xi_k) + m_k + n_k \]

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\(\rightarrow\) Sparse estimation to estimate MP biases on raw measurements
First idea

\[ z_k = h_k(\xi_k) + m_k + n_k \]

**Assumption:** 6 satellites \( \equiv \) 12 measurements
\[ \rightarrow \text{maybe 4 measurements suffer from MP} \]

\[ m_k = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \]
\[ \rightarrow \quad m_k = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \]

\[ \rightarrow \text{Sparse estimation to estimate MP biases on raw measurements} \]
Sparse Estimation
Sparse Regularization

Measurements
\[ \tilde{y}_k = \tilde{H}_k \theta_k + \tilde{n}_k \quad \text{with} \quad \tilde{H}_k \text{ low rank} \]
\[ \Rightarrow \text{need for appropriate regularization} \]

Assumption: \( \theta_k \) is sparse \( \Rightarrow \) minimize \( \| \theta_k \|_0 \)

LASSO
\[
\arg \min_{\theta_k} \left\{ \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| \theta_k \|_1 \right\}
\]

Weighted-\( \ell_1 \)
\[
\arg \min_{\theta_k} \left\{ \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| W_k \theta_k \|_1 \right\}
\]

---


Sparse Regularization

Measurements

\[ \mathbf{y}_k = \mathbf{H}_k \mathbf{\theta}_k + \mathbf{n}_k \quad \text{with} \quad \mathbf{H}_k \text{ low rank} \]

⇒ need for appropriate regularization

Assumption: \( \mathbf{\theta}_k \) is sparse \( \rightarrow \) minimize \( \| \mathbf{\theta}_k \|_0 \)

LASSO\(^8\)

\[
\arg\min_{\mathbf{\theta}_k} \left\{ \frac{1}{2} \| \mathbf{y}_k - \mathbf{H}_k \mathbf{\theta}_k \|_2^2 + \lambda_k \| \mathbf{\theta}_k \|_1 \right\}
\]

data-fidelity term

regularization term

Weighted-\(\ell_1\)\(^9\)

\[
\arg\min_{\mathbf{\theta}_k} \left\{ \frac{1}{2} \| \mathbf{y}_k - \mathbf{H}_k \mathbf{\theta}_k \|_2^2 + \lambda_k \| \mathbf{W}_k \mathbf{\theta}_k \|_1 \right\}
\]

---


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data-fidelity term

regularization term

Weighted-\( \ell_1 \)\(^9\)

\[
\arg \min_{\theta_k} \left\{ \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| W \theta_k \|_1 \right\}
\]

---


Sparse Regularization

Measurements

\[ \tilde{y}_k = \tilde{H}_k \theta_k + \tilde{n}_k \quad \text{with} \quad \tilde{H}_k \text{ low rank} \]

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LASSO\(^8\)

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Data-fidelity term

Regularization term

Weighted-\( \ell_1 \)\(^9\)

\[ \arg \min_{\theta_k} \left\{ \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| W_k \theta_k \|_1 \right\} \]


Sparse Regularization

Measurements

\[ \tilde{y}_k = \tilde{H}_k \theta_k + \tilde{n}_k \quad \text{with} \quad \tilde{H}_k \text{ low rank} \]

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\]

\[ \text{data-fidelity term} \quad \text{regularization term} \]

Weighted-\( \ell_1 \)\(^9\)

\[
\arg \min_{\theta_k} \left\{ \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| W_k \theta_k \|_1 \right\}
\]


Toy Example

\[
\begin{align*}
H &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \\
x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \\
W &= \begin{bmatrix} 0 & 0.2 \\
0 & 1 \end{bmatrix}
\end{align*}
\]

\[\hat{x}_\text{LASSO} = \arg \min_{x} \| Hx - y \|_2^2 + \lambda \| Wx \|_1 \]

\[\hat{x}_\text{W LASSO} = \arg \min_{x} \| Hx - y \|_2^2 + \lambda \| Wx \|_1 \]

\[\| x \|_2 = c \| x \|_1 = c \| Wx \|_1 \]

Thesis Defence Julien LESOUPLE TéSA, CNES, M3 Systems, IRIT, ISAE
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_2 = c \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_2 = c \]

\[ \|x\|_1 = c \]

\[ W = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix} \]

\[ \hat{x} \text{ LASSO} \]

\[ \hat{x} \text{ wLASSO} \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

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Toy Example

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Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \]
\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_1 = c \]

\[ \hat{x}_{\ell_2} \]

\[ \hat{x}_{\text{LASSO}} \]

\[ \hat{x}_{\text{W-LASSO}} \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_1 = c \]

\[ \hat{x}_{\ell_2} \]

\[ \hat{x}_{\text{LASSO}} \]

\[ \hat{x}_{\text{wLASSO}} \]
**Toy Example**

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_1 = c \]

\[ \hat{x}_{\ell_2} \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ y = Hx \]

\[ \|x\|_1 = c \]

\[ \|x\|_2 = c \]

\[ \hat{x}_\text{LASSO} \]

\[ \hat{x}_\ell_2 \]

\[ \hat{x}_\text{LASSO} \]

\[ \hat{x}_\ell_2 \]
Toy Example

\( H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad W = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix} \)

\[ \| Wx \|_1 = c \]

\[ y = Hx \]
Toy Example

\[ H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad W = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix} \]

Thesis Defence Julien LESOUPLE TéSA, CNES, M3 Systems, IRIT, ISAE
Toy Example

\[
H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad W = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}
\]
Application to Multipath Bias Estimation\textsuperscript{10,11}

Measurements

\[ z_k - h_k(\hat{\xi}_k|_{k-1}) = H_k(\xi_k - \hat{\xi}_k|_{k-1}) + m_k + n_k \]

\[ \iff y_k = H_kx_k + m_k + n_k \]

Assumption

\( m_k \) is sparse

Weighted-\( \ell_1 \)

\[ \arg\min \left\{ \frac{1}{2} \| y_k - H_kx_k - m_k \|_2^2 + \lambda_k \| W_km_k \|_1 \right\} \]


Application to Multipath Bias Estimation\textsuperscript{10,11}

Measurements

\[ z_k - h_k(\hat{\xi}_k|k-1) = H_k(\xi_k - \hat{\xi}_k|k-1) + m_k + n_k \]

\( \Leftrightarrow y_k = H_k x_k + m_k + n_k \)

Assumption

\( m_k \) is sparse

\[ \arg\min_{x_k,m_k} \left\{ \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1 \right\} \]


Application to Multipath Bias Estimation\textsuperscript{10,11}

Measurements

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\[ \iff y_k = H_kx_k + m_k + n_k \]

Assumption

\( m_k \) is sparse

Weighted-\( \ell_1 \)

\[ \arg \min_{x_k, m_k} \left\{ \frac{1}{2} \| y_k - H_kx_k - m_k \|^2_2 + \lambda_k \| W_k m_k \|_1 \right\} \]


Weights for Navigation

Weights related to *signal strengths* and *satellite elevations*\(^\text{12}\)

\[ C/N_0 \text{ [dBHz]} \]

\[ \text{Elev [deg]} \]

Additional Solutions

Avoid flickering in the estimation by **temporal smoothing**\(^\text{13}\)

- Total variation (Fused LASSO)\(^{\text{14}}\)

\[
\arg\min_{\theta_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| \theta_k \|_1 + \mu \| \theta_k - \hat{\theta}_{k-1} \|_1
\]

Robust estimation for the noise covariance matrix\(^{\text{15}}\)

- Danish method\(^{\text{16}}\)


Additional Solutions

Avoid flickering in the estimation by temporal smoothing\textsuperscript{13}

- Total variation (Fused LASSO)\textsuperscript{14}

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\arg\min_{\theta_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| \theta_k \|_1 + \mu \| \theta_k - \hat{\theta}_{k-1} \|_1
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Robust estimation for the noise covariance matrix\textsuperscript{15}

- Danish method\textsuperscript{16}


\textsuperscript{14} Robert Tibshirani, Michael Saunders, Saharon Rosset, Ji Zhu, and Keith Knight. “Sparsity and Smoothness via the Fused Lasso”. In: Journal of the Royal Statistical Society Series B (2005), pp. 91–108.


Additional Solutions

Avoid flickering in the estimation by temporal smoothing\textsuperscript{13}

- Total variation (Fused LASSO)\textsuperscript{14}

\[
\arg \min_{\theta_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|_2^2 + \lambda_k \| \theta_k \|_1 + \mu \| \theta_k - \hat{\theta}_{k-1} \|_1
\]

Robust estimation for the noise covariance matrix\textsuperscript{15}

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Avoid flickering in the estimation by temporal smoothing\(^{13}\)

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\]

Robust estimation for the noise covariance matrix\(^{15}\)

- Danish method\(^{16}\)


## Proposed Strategies

<table>
<thead>
<tr>
<th>Name</th>
<th>MP bias</th>
<th>Noise covariance</th>
</tr>
</thead>
<tbody>
<tr>
<td>EKF</td>
<td>$m_k = 0$</td>
<td>$\begin{bmatrix} \sigma^2_p l_{s_k} &amp; 0 \ 0 &amp; \sigma^2_r l_{s_k} \end{bmatrix}$</td>
</tr>
<tr>
<td>Weighted LASSO</td>
<td>Weighted-$\ell_1$</td>
<td>$\begin{bmatrix} \sigma^2_p l_{s_k} &amp; 0 \ 0 &amp; \sigma^2_r l_{s_k} \end{bmatrix}$</td>
</tr>
<tr>
<td>Fused LASSO</td>
<td>Weighted-$\ell_1$ and smoothing</td>
<td>$\begin{bmatrix} \sigma^2_p l_{s_k} &amp; 0 \ 0 &amp; \sigma^2_r l_{s_k} \end{bmatrix}$</td>
</tr>
<tr>
<td>Danish</td>
<td>$m_k = 0$</td>
<td>Danish method</td>
</tr>
<tr>
<td>Weighted LASSO + Danish</td>
<td>Weighted-$\ell_1$</td>
<td>Danish method</td>
</tr>
<tr>
<td>Fused LASSO + Danish</td>
<td>Weighted-$\ell_1$ and smoothing</td>
<td>Danish method</td>
</tr>
</tbody>
</table>
Experimental setup

- Ground truth: Novatel SPAN (GPS receiver Propak-V3 + inertial measurements unit IMAR)

- Measurements: Ublox AEK-4T
Trajectory
Local Results

Few MP
Local Results

Few MP

Legend
- Danish
- FDE
- Fused LASSO
- Fused LASSO + Danish
- Reference
- Standard EKF
- Weighted I2
- Weighted LASSO
- Weighted LASSO + Danish

Google Earth
Local Results

More MP
Local Results

More MP
Local Results

More robust methods

Legend
- FDE
- Reference
- Standard EKF
- Weighted l2
- Weighted LASSO

Google Earth
Local Results

More robust methods

Legend
- Danish
- FDE
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- Fused LASSO + Danish
- Reference
- Standard EKF
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- Weighted LASSO
- Weighted LASSO + Danish

Google Earth
Global Results: Planar Error

<table>
<thead>
<tr>
<th>Method</th>
<th>95-th %</th>
<th>Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>EKF</td>
<td>29.75m</td>
<td>148%</td>
</tr>
<tr>
<td>FDE</td>
<td>26.52m</td>
<td>121%</td>
</tr>
<tr>
<td>Weighted ℓ₂</td>
<td>20.67m</td>
<td>72%</td>
</tr>
<tr>
<td>Weighted ℓ₁</td>
<td>16.25m</td>
<td>35%</td>
</tr>
<tr>
<td>Fused ℓ₁</td>
<td>15.9 m</td>
<td>32%</td>
</tr>
<tr>
<td>Danish</td>
<td>13.75m</td>
<td>15%</td>
</tr>
<tr>
<td>Danish + ℓ₁</td>
<td>12.77m</td>
<td>6%</td>
</tr>
<tr>
<td>Danish + Fused ℓ₁</td>
<td>12.00m</td>
<td>0%</td>
</tr>
<tr>
<td>Ublox</td>
<td>13.67m</td>
<td>14%</td>
</tr>
</tbody>
</table>
Global Results: Altitude Error

<table>
<thead>
<tr>
<th>Method</th>
<th>95-th %</th>
<th>Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>EKF</td>
<td>39.34m</td>
<td>218%</td>
</tr>
<tr>
<td>FDE</td>
<td>39.38m</td>
<td>218%</td>
</tr>
<tr>
<td>Weighted $\ell_2$</td>
<td>27.63m</td>
<td>123%</td>
</tr>
<tr>
<td>Weighted $\ell_1$</td>
<td>13.95m</td>
<td>13%</td>
</tr>
<tr>
<td>Fused $\ell_1$</td>
<td>13.14 m</td>
<td>6%</td>
</tr>
<tr>
<td>Danish</td>
<td>13.75m</td>
<td>1%</td>
</tr>
<tr>
<td>Danish + Weighted $\ell_1$</td>
<td>11.34m</td>
<td>-8%</td>
</tr>
<tr>
<td>Danish + Fused $\ell_1$</td>
<td>12.38m</td>
<td>0%</td>
</tr>
<tr>
<td>Ublox</td>
<td>16.55m</td>
<td>34%</td>
</tr>
</tbody>
</table>
Tuning the hyperparameter

\[ \arg \min_{x_k, m_k} \left\{ \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1 \right\} \]
Tuning the hyperparameter

\[
\arg\min_{x_k, m_k} \left\{ \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1 \right\}
\]

Cross-validation

![Graph showing RMSE vs λ for different directions (East, North, Up)]
Bayesian Framework

Rewriting the problem

\[
\arg\min_{x_k, m_k} \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1
\]

\[
\Leftrightarrow \arg\max_{x_k, m_k} \exp \left( -\frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 \right) \exp (-\lambda_k \| W_k m_k \|_1)
\]

- Gaussian likelihood \( y_k | x_k, m_k \)
- Laplacian prior for \( m_k \)
- Missing
  - Prior for \( x_k \) (assuming independence between \( m_k \) and \( m_k \))
  - Hyperprior for \( \lambda_k \)
Bayesian Framework

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\[
\arg \min_{x_k, m_k} \frac{1}{2}\| \mathbf{y}_k - H_k x_k - m_k \|^2_2 + \lambda_k \| W_k m_k \|_1
\]

\[\iff\]

\[
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$$\arg\min_{x_k, m_k} \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1$$

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- Hyperprior for $\lambda_k$
Hierarchical Bayesian Model

Gaussian likelihood for $y_k$ (from model)

$$y_k | x_k, m_k \sim \mathcal{N}(y_k; H_k x_k + m_k, R_k)$$

Laplacian prior for $m_k$ (from model)

$$m_{i,k} \sim \mathcal{L}(m_{i,k}; 0, \frac{1}{\lambda_k w_{i,k}})$$

Gaussian prior for $x_k$ (from Kalman filter theory)

$$x_k \sim \mathcal{N}(x_k; 0, P_{k|k-1})$$

Jeffreys prior for $\lambda^2_k$ (non-informative prior)

$$\lambda^2_k \sim p(\lambda^2_k) \propto \frac{1}{\lambda^2_k}$$
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$$\lambda_k^2 \sim p(\lambda_k^2) \propto \frac{1}{\lambda_k^2}$$
Bayesian LASSO\textsuperscript{17}

Introduction of latent variable $\tau_k^2$

Posterior distribution

$$f(x_k, m_k, \tau_k^2, \lambda_k^2 | y_k) \propto f(y_k | x_k, m_k) f(x_k) f(m_k | \tau_k^2, \lambda_k^2) f(\tau_k^2 | \lambda_k^2) f(\lambda_k^2)$$

MAP estimator: Mode of this distribution

MMSE estimator: Mean of this distribution

$\rightarrow$ intractable

Bayesian LASSO\textsuperscript{17}

Introduction of latent variable $\tau^2_k$

Posterior distribution

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f(x_k, m_k, \tau^2_k, \lambda^2_k | y_k) \propto f(y_k | x_k, m_k) f(x_k) f(m_k | \tau^2_k, \lambda^2_k) f(\tau^2_k | \lambda^2_k) f(\lambda^2_k)
\]

MAP estimator: Mode of this distribution
MMSE estimator: Mean of this distribution

\[\rightarrow \text{intractable}\]

### MCMC Methods

Markov Chain Monte Carlo\(^{18}\) methods draw samples \(\theta^{(1)}, \theta^{(2)}, \ldots\) from posterior distribution of \(\theta\)

- posterior distribution is known up to a multiplicative constant
- samples \(y^{(t)}\) can be drawn from a proposal distribution
- set \(\theta^{(t)} = y^{(t)}\) with an appropriate acceptance probability

#### Gibbs sampling

- proposal distributions are the conditional distributions
- acceptance probability is 1

---

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Gibbs sampling

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- acceptance probability is 1

Conditional distributions

Latent variable

$\tau_{i,k}^2 | m_{i,k}, \lambda_k^2 \sim GIG(\tau_{i,k}^2; \frac{1}{2}, w_{i,k} \lambda_k^2, m_{i,k}^2)$

Multipath bias

$m_k | y_k, x_k, \tau_k^2 \sim \mathcal{N}(m_k; \mu_{m_k}, \Sigma_{m_k})$

State vector variation

$x_k | y_k, m_k \sim \mathcal{N}(x_k; K_k(y_k - m_k), P_{k|k})$

Hyperparameter

$\lambda_k^2 | \tau_k^2 \sim \mathcal{G} \left( \lambda_k^2; 2s_k, \frac{1}{2} \sum_{i=1}^{2s_k} w_{i,k}^2 \tau_{i,k}^2 \right)$

$\Sigma_{m_k} = \text{diag} \left( \frac{\sigma_{i,k}^2 \tau_{i,k}^2}{\sigma_{i,k}^2 + \tau_{i,k}^2} \right), \quad \mu_{m_k} = \text{diag} \left( \frac{\tau_{i,k}^2}{\sigma_{i,k}^2 + \tau_{i,k}^2} \right)(y_k - H_k x_k)$

$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k), \quad P_{k|k} = (I - K_k H_k) P_{k|k-1}$
Multipath Detection/Estimation: Synthetic Data

Simulation scenario

- 200 Monte Carlo iterations
- States and measurements generated by system equations
- Artificial (controlled) dynamic MP biases

Gibbs sampler

- 1000 iterations with a 100 burn-in period
- Convergence assessment\(^\text{19}\): PSRF < 1.2
- MMSE estimators: averages of generated samples

Multipath Detection/Estimation: Synthetic Data

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Multipath Detection/Estimation: Synthetic Data

Pseudoranges

- Bayesian algorithm
- wLASSO

Position

- EKF
- Bayesian algorithm
- wLASSO

Hyperparameter

$k = 25$ - 2 MP

Thesis Defence
Julien LESOUPLE
TéSA, CNES, M3 Systems, IRIT, ISAE

37/50
Multipath Detection/Estimation: Synthetic Data

Pseudoranges
Bayesian algorithm

Pseudorange rates
Bayesian algorithm

Position

Hyperparameter

EKF
Bayesian algorithm
wLASSO

Planar error [m]

Altitude error [m]

$k = 138$ : 0 MP
Multipath Detection/Estimation: Real Data

![Graphs showing multipath detection and estimation results for real data.](image)

- **Ublox**
- **EKF**
- **Bayesian algorithm**
- **wLASSO**

**Planar error [m]** vs. **Altitude error [m]**

**Legend:**
- Green: Ublox
- Red: EKF
- Orange: Bayesian algorithm
- Purple: wLASSO
Mixture Models
Main idea

Generalized problem

\[ z_k = h_k(\xi_k) + \underbrace{m_k + n_k}_{\nu_k} \]

\[ \Rightarrow \quad m_k \sim \mathcal{L}, \quad n_k \sim \mathcal{N}, \quad \nu_k \sim \mathcal{D} \]

Many distributions have been proposed

- Conditional Gaussian\(^{20}\)
- Gaussian mixtures\(^{21}\)
- Dirichlet process mixtures\(^{22}\)


Main idea

Generalized problem

\[ z_k = h_k(\xi_k) + m_k + n_k \]

\[ \nu_k \Rightarrow m_k \sim \mathcal{L}, \quad n_k \sim \mathcal{N} \]

\[ \nu_k \sim \mathcal{D} \]

Many distributions have been proposed

- Conditional Gaussian\textsuperscript{20}
- Gaussian mixtures\textsuperscript{21}
- Dirichlet process mixtures\textsuperscript{22}


Gaussian Mixtures

Model

\[ n_{i,k} \sim \sum_{\ell=1}^{M} \alpha_{i,\ell} \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \Leftrightarrow \begin{cases} P(c_{i,k} = \ell) = \alpha_{i,\ell} \\ n_{i,k} | c_{i,k} = \ell \sim \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \end{cases} \]

Estimation

- Expectation Maximization method\(^{23}\)

Gaussian Mixtures

Model

\[ n_{i,k} \sim \sum_{\ell=1}^{M} \alpha_{i,\ell} \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \Leftrightarrow \left\{ \begin{array}{l} P(c_{i,k} = \ell) = \alpha_{i,\ell} \\ n_{i,k} | c_{i,k} = \ell \sim \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \end{array} \right. \]

Estimation

- Expectation Maximization method\(^{23}\)

Hidden Markov Model

**Principle**
- Gaussian mixtures with dependence on previous state $k - 1$

**Model**

\[
n_{i,k} \sim \sum_{j=1}^{M} \alpha_{i,j} \mathcal{N}(n_{i,k}; \mu_{i,j}, \sigma_{i,j}^2)
\]

\[
\begin{align*}
P(c_{i,k} = \ell | c_{i,k-1} = m) &= (A_i)_{m,\ell} \\
P(c_{i,0} = \ell) &= (\Pi_i)_\ell \\
n_{i,k} | c_{i,k} = \ell &\sim \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2)
\end{align*}
\]

**Estimation**
- Baum-Welch method\(^{24}\)

---

Filters

Gaussian Mixtures

- Gaussian sum filter\(^{25}\): bank of Kalman filters for all modes of the mixtures

HMM

- Interacting Multiple Model\(^{26}\): bank of Kalman filters for all modes of the mixtures using approximations

Computing limitations

- Huge number of modes: \(M^{2^k}\)
- Limitation to a maximum of two mode changes

---


Some Experiments
Some Experiments
Cumulative Distribution Functions

- EKF
- Danish+Fused
- GM1
- GM2
- HMM1
- HMM2

Planar error [m] vs. %

Vertical error [m] vs. %
Conclusions and Future Works
Sparse Estimation

Advantages

▶ Joint detection/estimation of MP bias
▶ Only need raw measurements (RINEX) from any receiver
▶ Real-time formulation
▶ Can be combined to robust estimation

Drawback

▶ Hyperparameter tuning

Future work

▶ Other weighting matrices
▶ Other hyperparameter estimation: time-dependent, DOP-dependent, ...
▶ Fusion with other sensors/signals: multi constellation, multi frequency, vision, 5G, ...
Bayesian Estimation

Advantages

▶ No hyperparameter tuning
▶ Measures of uncertainties

Drawback

▶ Computationally intensive

Future work

▶ Assign different priors to multipath biases
▶ More informative priors for the hyperparameter
▶ Develop more efficient algorithms: SMC methods
Mixture Models

Advantages

- More flexibility
- Straightforward computations in the Gaussian case

Drawbacks

- Full solution computationally intensive: reduce the number of births and deaths
- Prior learning of the noise distribution

Future work

- Online estimation of the mixtures
- Optimize the mode configurations: MCMC, particular filters
- Combine sparse estimation and Gaussian mixtures
Sparsity in GNSS

Precise Point Positioning in urban environment

- Multi-frequency signals: instantaneous ambiguity resolution
- Use of sparse estimation to detect cycle slips

---

Sparsity in GNSS

Software Define Radio

- Versatile device
- Implement sparse estimation earlier in the receiver

ISAE Supaero
Sparsity in GNSS

Collaborative Positioning

- Increasing number of IoT sensors
- Stock and share data: cloud

Integrity

- Develop integrity criteria based on sparse estimation
- Spoofing and jamming detection/correction
- Authentication of the signals

---

Thanks for your attention!
Increasingly various GPS applications

GPS Signal

Extended Kalman Filter

Solving the Sparse Bias Problem

Discontinuities in Estimation

The $\ell_0$ Problem

Comparison with reweighted-$\ell_1$

Wavelet decomposition

Bayesian LASSO

Hierarchical Bayesian Model with MP indicator

Multipath Detection/Estimation: Hyperparameter evolution

Gaussian Mixtures
Increasingly various GPS applications

A. Brzezinski et al., “Geodetic and Geodynamic Studies at Department of Geodesy and Geodetic Astronomy Wut”, in Reports on Geodesy and Geoinformatics vol. 100, March 2016, pp.165-200

Marielle Mayo, “GNSS-R Signaux réfléchis”, in Géomètre n°2123, March 2015, pp.46-49
GPS Signal

- **L1 Carrier**
- **C/A Code**
- **Navigation Message**
- **P Code**

Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)
State propagation $\xi_k \in \mathbb{R}^8$

Hypothesis: random walk

$$\xi_k = F_k \xi_{k-1} + u_k \quad \text{with} \quad F_k \text{ known}$$

$$u_k \sim \mathcal{N}(n_k; 0, Q_k)$$

**EKF** = Kalman Filter + Linearization

Kalman predictions

$$\hat{\xi}_{k|k-1} = F_k \hat{\xi}_{k-1|k-1}$$
$$P_{k|k-1} = F_k P_{k-1|k-1} F_k + Q_k$$

→ Linearization point

Kalman updates

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)$$

$$\hat{\xi}_k = \hat{\xi}_{k|k-1} + K_k (z_k - h_k(\hat{\xi}_{k|k-1}) - m_k)$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$
Tuning the EKF

**State Evolution**

\[ F_{k-1} = \begin{bmatrix} I_4 & \Delta t_k \\ 0 & I_4 \end{bmatrix} \]

**State Covariance**

\[ Q_{k-1} = \begin{bmatrix} S_a \frac{\Delta t_{k-1}^3}{3} I_3 \\ 0_{1\times3} \\ S_a \frac{\Delta t_{k-1}^2}{2} I_3 \\ 0_{1\times3} \end{bmatrix} 
+ c^2 \begin{bmatrix} S_b \Delta t_{k-1} + S_d \frac{\Delta t_{k-1}^2}{3} \\ 0_{3\times1} \\ S_a \frac{\Delta t_{k-1}^2}{2} I_3 \\ 0_{1\times3} \end{bmatrix} 
+ \begin{bmatrix} 0_{3\times1} \\ S_a \Delta t_{k-1} I_3 \\ 0_{3\times1} \\ c^2 \left( S_d \frac{\Delta t_{k-1}^2}{2} \right) \end{bmatrix} \]
Solving the Sparse Bias Problem

Measurements:

\[ y_k = H_k x_k + m_k + n_k \]

Profile likelihood:

\[ x_k = (H_k^T H_k)^{-1} H_k^T (y_k - m_k) \]

\[
\arg\min_{x_k, m_k} \frac{1}{2} \| y_k - H_k x_k - m_k \|^2_2 + \lambda_k \| W_k m_k \|_1
\]

\[
\arg\min_{m_k} \frac{1}{2} \| y_k - H_k (H_k^T H_k)^{-1} H_k^T (y_k - m_k) - m_k \|^2_2 + \lambda_k \| W_k m_k \|_1
\]

\[
\arg\min_{m_k} \frac{1}{2} \| (I - P_k) y_k - (I - P_k) W_k^{-1} W_k m_k \|^2_2 + \lambda_k \| W_k m_k \|_1
\]

\[
\arg\min_{\theta_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k \theta_k \|^2_2 + \lambda_k \| \theta_k \|_1 \rightarrow \text{LASSO problem}
\]
Discontinuities in Estimation

Satellite 13

Pseudospeed Bias [m.s⁻¹]

C/N₀ [dBHz]

Snapshots
Non-convex Problem

$$\arg \min_{\theta_k} \frac{1}{2} \left\| \tilde{y}_k - \tilde{H}_k \theta_k \right\|_2^2 + \lambda_k \left\| \theta_k \right\|_0$$
Results
Results
Results
Rewighted-$\ell_1$

Initialize $W$

for $\ell = 0, \ldots, \ell_{\text{max}}$ do

Solve $\theta^{(\ell)} = \arg\min_{\theta \in \mathbb{R}^n} \frac{1}{2} \|\tilde{y} - \tilde{H}\theta\|_2^2 + \lambda \|W^{(\ell)}\theta\|_1$

Update weights

for $i = 1, \ldots, n$ do

$w_i^{(\ell+1)} = \frac{1}{\theta_i^{(\ell)} + \epsilon}$

end for

end for
Results
Results
Results
Wavelets decomposition

\[ \text{arg min}_{m_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k m_k \|_2^2 + \lambda_k \| W_k m_k \|_1 \]

\[ \text{arg min}_{m_k} \frac{1}{2} \| \tilde{y}_k - \tilde{H}_k m_k \|_2^2 + \lambda_k \| \psi_k m_k \|_1 \] (1)

(2)

Collaboration with Universidad Industrial de Santander (Columbia)
Results
Results
Bayesian LASSO

Completion (marginalization trick)

\[
\frac{w_{i,k} \lambda_k}{2} \exp (-w_{i,k} \lambda_k | m_{i,k} |) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi s}} \exp \left( -\frac{m_{i,k}^2}{2s} \right) \frac{w_{i,k}^2 \lambda_k^2}{2} \exp \left( -\frac{w_{i,k}^2 \lambda_k^2 s}{2} \right) ds
\]

\[
m_{i,k} | \lambda_k^2 \sim \mathcal{L} \left( m_{i,k}; 0, \frac{1}{\lambda_k w_{i,k}} \right) \iff \exists \tau_{i,k}^2,
\begin{cases}
  m_{k} | \tau_{i,k}^2 \sim \mathcal{N}(m_{i,k}; 0, \tau_{i,k}^2) \\
  \tau_{i,k}^2 | \lambda_k^2 \sim \mathcal{E} \left( \tau_{i,k}^2; \frac{2}{\lambda_k^2 w_{i,k}} \right)
\end{cases}
\]

Posterior distribution

\[
f(\mathbf{x}_k, \mathbf{m}_k, \tau_k^2, \lambda_k^2 | \mathbf{y}_k) \propto f(\mathbf{y}_k | \mathbf{x}_k, \mathbf{m}_k) f(\mathbf{x}_k) f(\mathbf{m}_k | \tau_k^2, \lambda_k^2) f(\tau_k^2 | \lambda_k^2) f(\lambda_k^2)
\]

\{\text{likelihood} \quad \text{priors} \quad \text{hyperprior}\}
Hierarchical Bayesian Model with MP indicator

\[ y_k | x_k, m_k, \sim \mathcal{N}(y_k; H_k x_k + m_k, R_k) \]

\[ x_k \sim \mathcal{N}(x_k; 0, P_{k|k-1}) \]

\[ m_{i,k} | b_{i,k}, \tau_{i,k}^2 \sim \begin{cases} 
\delta(m_{i,k}) & \text{if } b_{i,k} = 0 \\
\mathcal{N}(m_{i,k}; 0, \tau_{i,k}^2) & \text{if } b_{i,k} = 1 
\end{cases}, i = 1, \ldots, 2s_k \]

\[ \tau_{i,k}^2 | \lambda_k^2 \sim \mathcal{E}\left(\tau_{i,k}^2; \frac{2}{\lambda_k^2 w_{i,k}^2}\right), i = 1, \ldots, 2s_k \]

\[ b_{i,k} | p_k \sim \mathcal{B}(b_{i,k}; p_k), i = 1, \ldots, 2s_k \]

\[ p_k \sim \mathcal{U}_{[0,1]}(p_k) \]

\[ f(\lambda_k^2) \propto \frac{1}{\lambda_k^2} \]
Conditional distributions with MP indicator

Latent variable
\[ \tau_{i,k}^2 | m_{i,k}, \lambda_k^2, b_{i,k} \begin{cases} \mathcal{E} \left( \frac{\tau_{i,k}^2}{w_{i,k}^2 \lambda_k^2} \right) & \text{if } b_{i,k} = 0 \\ \mathcal{IG} \left( \tau_{i,k}^2 ; \frac{1}{2}, w_{i,k}^2 \lambda_k^2, m_{i,k}^2 \right) & \text{if } b_{i,k} = 1 \end{cases} \]

Multipath indicator
\[ b_{i,k} | y_{i,k}, x_k, \tau_{i,k}^2, p_k = \mathcal{B} \left( b_{i,k} \mid \frac{v_{i,k}}{u_{i,k} + v_{i,k}} \right) \]

Multipath bias
\[ m_k | y_k, x_k, \tau_k^2 \sim \begin{cases} \delta(m_{i,j}) & \text{if } b_{i,k} = 0 \\ \mathcal{N}(\mu_{m,i,k}, \sigma_{m,i,k}^2) & \text{if } b_{i,k} = 1 \end{cases} \]

State vector variation
\[ x_k | y_k, m_k \sim \mathcal{N}(x_k; K_k(y_k - m_k), P_{k|k}) \]

Hyperparameter \( \lambda_k \)
\[ \lambda_k^2 | \tau_k^2 \sim \mathcal{G} \left( \lambda_k^2 ; 2s_k, \frac{1}{2} \sum_{i=1}^{2s_k} w_{i,k}^2 \tau_{i,k}^2 \right) \]

Hyperparameter \( p_k \)
\[ f(p_k | b_k) = \mathcal{B}e \left( p_k ; \| b_k \|_0 + 1, 2s_k - \| b_k \|_0 + 1 \right) \]

\[ u_{i,k} = (1 - p_k), \quad v_{i,k} = p_k \sqrt{\frac{\sigma_{m,i,k}^2}{\tau_{i,k}^2}} \exp \left( \frac{\mu_{m,i,k}^2}{2 \sigma_{m,i,k}^2} \right) \]
Evolution of the hyperparameter $\lambda_k$ and the number of multipaths

- $\lambda_k$: Hyperparameter evolution over $k$
- Number of multipaths: Variation with $k$ from 0 to 140
Gaussian Mixtures

Parameters: \( A_i, \Pi_i, \mu_i, \sigma_i^2 \)

Used online

3 modes per satellite

Modes evolution estimated before \((C/N_0 \text{ values})\)

\( A_i \) and \( \Pi_i \) are the proportions
\( \mu_i, \sigma_i^2 \) estimated via residuals

Particle Filter

Need to learn the distributions

\( M \) modes per measurement

Modes evolution estimated after \((\text{MAP estimator})\)

\( A_i, \Pi_i, \mu_i, \sigma_i^2 \) are estimated via Baum-Welch

Bank of Kalman filters