Smooth Bias Estimation for Multipath Mitigation Using Sparse Estimation - Supplementary materials

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Abstract

This document contains supplementary materials associated with the conference paper [1].

1 Introduction

The aim of the paper [1] was to introduce a smooth sparse estimation method of the multipath (MP) biases in Global Navigation Satellite Systems (GNSS) measurements (pseudoranges and Doppler). To do so, we supposed the receiver ate time instant k has s_k satellites in view, and thus has access to $2s_k$ measurements (s_k pseudoranges and s_k pseudorange rates) which can be expressed as

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{m}_k + \boldsymbol{n}_k \tag{1}$$

where $\boldsymbol{y}_k = (y_{i,k})_{i=1,\ldots,2s_k} \in \mathbb{R}^{2s_k}$ is a vector containing the differences between the measurements and their estimates, $\boldsymbol{H}_k \in \mathbb{R}^{2s_k \times 8}$ is the Jacobian matrix of the problem, and $\boldsymbol{n}_k$ is an additive white Gaussian noise with covariance matrix $\boldsymbol{R}_k$. In the paper, it was proposed to solve the two problems

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2} \|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_1 \tag{2}$$

and

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2} \|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_2 \tag{3}$$

where

- $\boldsymbol{P}_k = \boldsymbol{H}_k(\boldsymbol{H}_k^T \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T \in \mathbb{R}^{2s_k \times 2s_k}$ is projection matrix on the subspace spanned by $\boldsymbol{H}_k$
- $\tilde{\boldsymbol{y}}_k = (I_{2s_k} - \boldsymbol{P}_k)\boldsymbol{y}_k \in \mathbb{R}^{2s_k}$ are the projected measurements on the orthogonal space of $\boldsymbol{H}_k$
- $\tilde{\boldsymbol{H}}_k = (I_{2s_k} - \boldsymbol{P}_k)\boldsymbol{W}_k^{-1} \in \mathbb{R}^{2s_k \times 2s_k}$ is the weighted projection matrix on the orthogonal space of $\boldsymbol{H}_k$
- $\boldsymbol{\theta}_k = \boldsymbol{W}_k \boldsymbol{m}_k \in \mathbb{R}^{2s_k}$ are the weighted multipath biases

- $\hat{\boldsymbol{\theta}}_k \in \mathbb{R}^{2s_k}$ are the weighted multipath biases estimated at time instant $k-1$
- $\boldsymbol{W}_k \in \mathbb{R}^{2s_k \times 2s_k}$ is the weighting matrix
- $\mathcal{S}_k$ contains the set of satellites jointly visible at time instants k and $k-1$
- $\|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_{1,\mathcal{S}_k} = \sum_{i\in\mathcal{S}_k} |\theta_{i,k} - \hat{\theta}_{i,k-1}|$ is the ℓ_1-norm of $\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}$ restricted to the satellites presents at time instant $k-1$ and k
- λ_k and μ_k are two regularization parameters.

Next section will present how to solve problems (2) and (3).

2 Solution of the ℓ_2-smoothing regularization

For problem (3), one can define

$$\boldsymbol{\Delta}_k = \mathrm{diag}(i \in \mathcal{S}_k), \tag{4}$$

that is the diagonal matrix whose i-th entry is 1 if the corresponding satellite was visible at the previous time instant, and 0 if it was not, then the problem (3) can be rewritten

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \mu_k\|\boldsymbol{\Delta}_k(\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1})\|_2. \tag{5}$$

Let's develop this equation, noticing that $\boldsymbol{\Delta}_k^T = \boldsymbol{\Delta}_k$ and $\boldsymbol{\Delta}_k^2 = \boldsymbol{\Delta}_k$,

$$\begin{aligned}
&\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \mu_k\|\boldsymbol{\Delta}_k(\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1})\|_2^2 \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}(\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k)^T(\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k) + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \frac{1}{2}2\mu_k(\boldsymbol{\Delta}_k(\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}))^T(\boldsymbol{\Delta}_k(\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1})) \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\left(\boldsymbol{\theta}_k^T\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k - 2\tilde{\boldsymbol{y}}_k^T\tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k + 2\mu_k\boldsymbol{\theta}_k^T\boldsymbol{\Delta}_k\boldsymbol{\theta}_k - 4\mu_k\hat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{\Delta}_k\boldsymbol{\theta}_k\right) + \lambda_k\|\boldsymbol{\theta}_k\|_1 \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\left(\boldsymbol{\theta}_k^T(\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k)\boldsymbol{\theta}_k - 2(\tilde{\boldsymbol{y}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\hat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{\Delta}_k)\boldsymbol{\theta}_k\right) + \lambda_k\|\boldsymbol{\theta}_k\|_1
\end{aligned} \tag{6}$$

define $\boldsymbol{A}_k$ such that $\boldsymbol{A}_k^T\boldsymbol{A}_k = \tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k$ (which exists as $\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k$ has positive or null singular values given its expression), and assume the matrix $\boldsymbol{A}_k$ is invertible[1], then

$$\begin{aligned}
&\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\left(\boldsymbol{\theta}_k^T(\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\boldsymbol{\Delta}_k)\boldsymbol{\theta}_k - 2(\tilde{\boldsymbol{y}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\hat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{\Delta}_k)\boldsymbol{\theta}_k\right) + \lambda_k\|\boldsymbol{\theta}_k\|_1 \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\left(\boldsymbol{\theta}_k^T\boldsymbol{A}_k^T\boldsymbol{A}_k\boldsymbol{\theta}_k - 2(\tilde{\boldsymbol{y}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\hat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{\Delta}_k)\boldsymbol{A}_k^{-1}\boldsymbol{A}_k\boldsymbol{\theta}_k\right) + \lambda_k\|\boldsymbol{\theta}_k\|_1 \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|((\tilde{\boldsymbol{y}}_k^T\tilde{\boldsymbol{H}}_k + 2\mu_k\hat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{\Delta}_k)\boldsymbol{A}_k^{-1})^T - \boldsymbol{A}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 \\
\Leftrightarrow &\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2}\|(\boldsymbol{A}_k^T)^{-1}(\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{y}}_k + 2\mu_k\boldsymbol{\Delta}_k\hat{\boldsymbol{\theta}}_{k-1}) - \boldsymbol{A}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1
\end{aligned} \tag{7}$$

where one can recognize a LASSO problem with measurements $(\boldsymbol{A}_k^T)^{-1}(\tilde{\boldsymbol{H}}_k^T\tilde{\boldsymbol{y}}_k + 2\mu_k\boldsymbol{\Delta}_k\hat{\boldsymbol{\theta}}_{k-1})$, matrix $\boldsymbol{A}_k$ and unknown $\boldsymbol{\theta}_k$, and can use one of the method provided in [1] to solve it.

[1] This has always been the case in our experiments, but if not one has to treat separately satellites that where visible at time instant $k-1$ (as the matrix $\boldsymbol{A_k}$ will be invertible for them as it consists in diagonal loading) and those who were note

3 Solution of the ℓ_2-smoothing regularization

The problem to be solved is (2), namely

$$\underset{\boldsymbol{\theta}_k \in \mathbb{R}^{s_k}}{\arg\min} \frac{1}{2} \|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{\theta}_k\|_1 + \mu_k \|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_1.$$

First let's solve this problem when $\boldsymbol{\theta}$ is of dimension 1. The problem becomes, where $\mu_k = 0$ if the corresponding satellite weren't visible at time instant $k-1$ and $\mu_k > 0$ if it were,

$$\underset{\theta_k \in \mathbb{R}}{\arg\min} \frac{1}{2} \|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{h}}_k \theta_k\|_2^2 + \lambda_k |\theta_k| + \mu_k |\theta_k - \hat{\theta}_{k-1}| \tag{8}$$

whose generalized gradient is

$$\nabla \left(\frac{1}{2} (\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k^2 - 2 \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k \theta_k) \right) + \partial(\lambda_k |\theta_k|) + \partial(\mu_k |\theta_k - \hat{\theta}_{k-1}|) = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k \partial(|\theta_k|) + \mu_k \partial(|\theta_k - \hat{\theta}_k|) \tag{9}$$

where

$$\partial(|x|) = \begin{cases} \text{sign}(x) & \text{if } x \neq 0 \\ [-1, 1] & \text{if } x = 0 \end{cases}. \tag{10}$$

Hence there are $3^2 = 9$ according to what is inside each absolute value ($< 0, > 0$ or $= 0$).

- If $\theta_k > 0$ and $\theta_k - \hat{\theta}_{k-1} > 0$

We have

$$\nabla = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k + \mu_k \tag{11}$$

which equals 0 when

$$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k - \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k} \tag{12}$$

and the assumptions ($\theta_k > 0$ and $\theta_k - \hat{\theta}_{k-1} > 0$) give

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \lambda_k + \mu_k \tag{13}$$

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} + \lambda_k + \mu_k. \tag{14}$$

- If $\theta_k > 0$ and $\theta_k - \hat{\theta}_{k-1} < 0$

  We have

$$\nabla = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k - \mu_k \tag{15}$$

which equals 0 when

$$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k + \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k} \tag{16}$$

and the assumptions ($\theta_k > 0$ and $\theta_k - \hat{\theta}_{k-1} < 0$) give

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \lambda_k - \mu_k \tag{17}$$

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} + \lambda_k - \mu_k. \tag{18}$$

- If $\theta_k < 0$ and $\theta_k - \hat{\theta}_{k-1} > 0$

We have

$$\nabla = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k + \mu_k \tag{19}$$

which equals 0 when

$$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k - \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k} \tag{20}$$

and the assumptions ($\theta_k < 0$ and $\theta_k - \hat{\theta}_{k-1} > 0$) give

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < -\lambda_k + \mu_k \tag{21}$$

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} - \lambda_k + \mu_k. \tag{22}$$

- If $\theta_k < 0$ and $\theta_k - \hat{\theta}_{k-1} < 0$

We have

$$\nabla = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k - \mu_k \tag{23}$$

which equals 0 when

$$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k + \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k} \tag{24}$$

and the assumptions ($\theta_k < 0$ and $\theta_k - \hat{\theta}_{k-1} < 0$) give

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < -\lambda_k - \mu_k \tag{25}$$

$$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} - \lambda_k - \mu_k. \tag{26}$$

In cases where one of the absolute values is null, we can treat jointly the different signs of the other one

- If $\theta_k = 0$ and $\theta_k - \hat{\theta}_{k-1} \neq 0$

We have

$$\nabla = -\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k[-1, 1] + \mu_k \text{sign}(-\hat{\theta}_{k-1}) \tag{27}$$

and we solve

$$\begin{aligned} & 0 \in -\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k[-1, 1] + \mu_k \text{sign}(-\hat{\theta}_{k-1}) \\ \Leftrightarrow & \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \mu_k \text{sign}(\hat{\theta}_{k-1}) \in [-\lambda_k, \lambda_k] \\ \Leftrightarrow & |\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \mu_k \text{sign}(\hat{\theta}_{k-1})| \leq \lambda_k. \end{aligned} \tag{28}$$

- If $\theta_k \neq 0$ and $\theta_k - \hat{\theta}_{k-1} = 0$

We have $\theta_k = \hat{\theta}_{k-1}$ and

$$\nabla = \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k \text{sign}(\hat{\theta}_{k-1}) + \mu_k[-1, 1] \tag{29}$$

and we solve

$$\begin{aligned} & 0 \in \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k - \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k \text{sign}(\hat{\theta}_{k-1}) + \mu_k[-1, 1] \\ \Leftrightarrow & - \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k + \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k \text{sign}(\hat{\theta}_{k-1}) \in [-\mu_k, \mu_k] \\ \Leftrightarrow & | - \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k + \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k \text{sign}(\hat{\theta}_{k-1})| \leq \mu_k. \end{aligned} \tag{30}$$

Conditions	Solution
$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \lambda_k + \mu_k$ and $\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} + \lambda_k + \mu_k$	$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k - \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k}$
$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \lambda_k - \mu_k$ and $\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} + \lambda_k - \mu_k$	$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k + \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k}$
$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < -\lambda_k + \mu_k$ and $\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k > \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} - \lambda_k + \mu_k$	$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k - \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k}$
$\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < -\lambda_k - \mu_k$ and $\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k < \tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \hat{\theta}_{k-1} - \lambda_k - \mu_k$	$\theta_k^* = \frac{\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k + \mu_k}{\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k}$
$\lvert \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \mu_k \mathrm{sign}(\hat{\theta}_{k-1}) \rvert \leq \lambda_k$ and $\hat{\theta}_{k-1} \neq 0$	${\theta_k}^* = 0$
$\lvert -\tilde{\boldsymbol{h}}_k^T \tilde{\boldsymbol{h}}_k \theta_k + \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k - \lambda_k \mathrm{sign}(\hat{\theta}_{k-1}) \rvert \leq \mu_k$ and $\hat{\theta}_{k-1} \neq 0$	${\theta_k}^* = \hat{\theta}_{k-1}$
$\lvert \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k \rvert \leq \lambda_k + \mu_k$ and $\hat{\theta}_{k-1} = 0$	${\theta_k}^* = 0$

Table 1: Solution of the 1D problem.

- If $\theta_k = 0$ and $\theta_k - \hat{\theta}_{k-1} = 0$

We have $\theta_k = \hat{\theta}_{k-1}$ and

$$\nabla = -\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k[-1,1] + \mu_k[-1,1] \tag{31}$$

and we solve

$$\begin{aligned} & 0 \in -\tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k + \lambda_k[-1,1] + \mu_k[-1,1] \\ \Leftrightarrow & \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k \in [-(\lambda_k + \mu_k), \lambda_k + mu_k] \\ \Leftrightarrow & \lvert \tilde{\boldsymbol{y}}_k^T \tilde{\boldsymbol{h}}_k \rvert \leq \lambda_k + \mu_k. \end{aligned} \tag{32}$$

All these results are gathered in Table 1.

For the multidimensional case, one can apply the same strategy as done for the shooting algorithm in [2] and update each coordinate one at a time until convergence.

References

[1] J. Lesouple, F. Barbiero, F. Faurie, M. Sahmoudi, and J.-Y. Tourneret, "Smooth Bias Estimation for Multipath Mitigation Using Sparse Estimation," in *Proc. IEEE Int. Conf. on Inf. Fusion (FUSION)*, Cambridge, UK, July 2018, pp. xx–xx.

[2] G. V. Pendse, "A tutorial on the LASSO and the shooting algorithm," *Harvard Medical School*, vol. 13, Feb. 2011.