An EM Approach for GNSS Parameters of Interest Estimation Under Constant Modulus Interference

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***Abstract*—Interferences are an important threat for applications relying on Global Navigation Satellite Systems (GNSS). Interferences degrade GNSS performance, and can lead to denial of service. The most notable intentional interference family is characterized by its constant envelope, e.g. chirp and tone interferences. Due to its simple structure, the space to search the interference contribution yields to complex circles, allowing the introduction of some latent variables related to those circles. In order to mitigate the interference effect, we compute the maximum likelihood estimator of the parameters of interest (time delay and Doppler shift) in presence of those latent variables. Thus, we resort to the Expectation Maximization algorithm which has already been proved to be efficient in such cases. Experiments conducted on synthetic signals highlight the efficiency of the proposed algorithm.**

***Index Terms*—Maximum likelihood, Expectation maximization, GNSS, Interference, CEM.**

I. Introduction

Global Navigation Satellite Systems (GNSS) [1] are widely used not only in applications requiring navigation and timing information, but also in domains such as Earth observation, attitude estimation or space weather characterization. Therefore, reliable position, navigation and timing information is fundamental, especially for critical application such as intelligent transportation systems or autonomous unmanned ground/air vehicles. GNSS have become the main source of positioning data, however they were originally designed to operate in clear sky nominal conditions, and their performance clearly degrades under harsh environments. For instance, phenomenon such as multipath (reflections) [2], spoofing and interferences (intentional or unintentional) are the most challenging ones, being a key issue in safety-critical scenarios [3], such as civilian aviation [4]. These effects have been reported in the state-of-the-art, and several interference mitigation countermeasures have already been proposed [5]. In the time domain are two widespread methods: 1) pulse blanking [6], where samples of the incoming signal corresponding to a power level higher than a predefined threshold are set to zero, can be performed to reduce the impact of pulsed interference, and 2) adaptive notch filtering [7], where the instantaneous frequency of the jamming signal is continuously estimated through a recurrence equation in the time domain (hence no transformation in the frequency domain is required), and the corresponding frequency is then filtered out from the incoming signal. Similar to the pulse blanking, the signal can be projected in other domains, such as the frequency domain through the Discrete Fourier Transform (DFT), and a threshold can be applied to set to zero suspicious elements. Another transformation interesting for interference mitigation is the Karhunen-Loeve Transform (KLT) [8], [9], based on the incoming signal autocorrelation eigen values and vectors.

In this article, we propose a new way to reduce interference characterized by constant envelope (CE). This family of interferences represents one of the most important families of intentional interferences that have been detected in the state of the art. Examples of those signals are pure tones or time-varying tones, well known as chirped signals. Due to the constant modulus property, the space to search the interference contribution at the receiver yields to complex circles. In order to characterize these circles, some latent variables can be defined. The contribution of this article yields to compute the maximum likelihood estimator (MLE) of the parameters of interest, i.e., the time-delay and Doppler shift in the presence of those latent variables. In order to compute the MLE, we resort to the Expectation-Maximization (EM) algorithm which has already proven to be asymptotically efficient in such scenarios. For instance, this algorithm has been proven to be very effective for N-hypersphere estimation [10]. In order to evaluate the performance of the proposed algorithm, we compare our solution with respect to the theoretical limits of delay and Doppler shift estimation given by Cramér-Rao bound considering the well-specified [11] and misspecified [12] conditional signal models.

The article is organized as follows: Section II introduces the GNSS received signal model when a constant envelope band-limited interfering signal is attacking the receiver. Section III details the proposed EM algorithm used for the interference mitigation under the constant envelope hypothesis. Simulations results presented in Section IV validates the performance of the proposed method for two synthetic signal scenarios. Finally, conclusion are drawn in Section V.

II. Signal model

In this article, we consider a band-limited signal $s(t)$, with bandwidth B, transmitted over a carrier frequency f_c and traveling at the speed of light c, from a GNSS satellite to a receiver. The transmitter and receiver are assumed to be in uniform linear motion such as the distance can be modeled by a first order $d-v$ distance-velocity model [13]. At the receiver, a narrow-band signal model is assumed and the received signal $x(t)$ at the output of the receiver's Hilbert filter can be approximated by [11], [14]

$$x(t)=\rho e^{j\phi}s(t-\tau)e^{-j2\pi f_c b(t-\tau)}+n(t) \quad (1)$$

where ρ and ϕ are the amplitude and phase of the complex coefficient $\alpha=\rho e^{j\varphi}\in\mathbb{C}$ induced by the propagation characteristics, $\tau=d/c$ is the unknown propagation delay, $b=v/c$ is the unknown Doppler coefficient and $n(t)$ is a zero-mean white complex circular Gaussian noise. Assume an interfering signal $I(t)$, unknown and bandlimited within the frequency band of interest, is also arriving at the receiver. Then the signal x becomes

$$x(t)=\alpha s(t-\tau)e^{-j2\pi f_c b(t-\tau)}+I(t)+n(t). \quad (2)$$

Considering the acquisition of $N=N_2-N_1+1$ samples at the sampling frequency $F_s=B=1/T_s$, and assuming that the observation window $[N_1T_s,N_2T_s]$ is short enough to consider constant amplitude, delay and Doppler shift, the discrete signal model yields to

$$\boldsymbol{x}=\alpha\boldsymbol{\mu}(\boldsymbol{\eta})+\boldsymbol{I}+\boldsymbol{n} \quad (3)$$

where $\boldsymbol{\mu}(\boldsymbol{\eta})=\left[\ldots,s(kTs-\tau)e^{-j2\pi f_c b(kT_s-\tau)},\ldots\right]^T\in\mathbb{C}^N$ with $\boldsymbol{\eta}=(\tau,b)$ and $k\in(N_1,\ldots,N_2)$, $\boldsymbol{I}=[\ldots,I(kT_s),\ldots]^T\in\mathbb{C}^N$ and $\boldsymbol{n}=[\ldots,n(kT_s),\ldots]^T\sim\mathcal{CN}(0,\sigma^2 I_N)$. Moreover, under CE interference, all the components of the vector $\boldsymbol{I}$ have the same modulus $A>0$. We therefore propose the parametrization

$$\boldsymbol{I}=A\tilde{\boldsymbol{I}} \quad (4)$$

such that $\tilde{\boldsymbol{I}}=\begin{bmatrix}\tilde{I}_1 & \ldots & \tilde{I}_N\end{bmatrix}^T$ with $|\tilde{I}_i|=1$. In other words, each component of the vector $\tilde{\boldsymbol{I}}$ belongs to the complex unit circle. Hence, there exists $\boldsymbol{\theta}=\begin{bmatrix}\theta_1 & \ldots & \theta_N\end{bmatrix}^T\in[0,2\pi)^N$ such that

$$\forall i=1\ldots,N,\quad \tilde{I}_i=e^{j\theta_i}. \quad (5)$$

The resulting problem has the following conditional likelihood

$$p(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{\varepsilon})=\frac{1}{\pi^N\sigma^{2N}}e^{-\frac{1}{\sigma^2}\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})-A\tilde{\boldsymbol{I}}\right)^H\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})-A\tilde{\boldsymbol{I}}\right)} \quad (6)$$

where $\boldsymbol{\varepsilon}=\{\boldsymbol{\eta}^T,\rho,\varphi,A,\sigma^2\}$ is the vector gathering the parameters of interest. The conditional likelihood (6) can be marginalized w.r.t. $\boldsymbol{\theta}$, leading

$$\begin{aligned}p(\boldsymbol{x}|\boldsymbol{\varepsilon})=&\frac{1}{(\pi\sigma^2)^N}e^{-\frac{1}{\sigma^2}(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))^H(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))}e^{-\frac{A^2N}{\sigma^2}}\\&\times\prod_{i=1}^{N}I_0\left(\frac{2A}{\sigma^2}\left|x_i-\alpha\mu_i(\boldsymbol{\eta})\right|\right).\end{aligned} \quad (7)$$

However, one cannot derive closed-form expressions for the maximum likelihood estimators of the parameters in $\boldsymbol{\varepsilon}$ from this expression. One way to bypass this limit is to consider the variables in $\boldsymbol{\theta}$ as missing variables. The EM algorithm [15] can be handy to evaluate the maximum likelihood estimator of parameters when missing variables appear in the estimation framework.

III. EM approach for interference mitigation under the CE hypothesis

A. Complete likelihood

The complete likelihood of the parameters $\boldsymbol{\varepsilon}$ given the observations $\boldsymbol{x}$ and missing variables $\boldsymbol{\theta}$ can be expressed as

$$\begin{aligned}\mathcal{L}_c(\boldsymbol{\varepsilon};\boldsymbol{x},\boldsymbol{\theta})&=p(\boldsymbol{x},\boldsymbol{\theta}|\boldsymbol{\varepsilon})\\&=p(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{\varepsilon})p(\boldsymbol{\theta}).\end{aligned}$$

where $p(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{\varepsilon})$ is given in (6). In absence of hypothesis on the interfering signal, we can take independent uniform priors on $[0,2\pi)$ for the θ_i's

$$p(\boldsymbol{\theta})=\prod_{i=1}^{N}\frac{1}{2\pi}1_{[0,2\pi)}(\theta_i) \quad (8)$$

where $1_{[0,2\pi)}(.)$ is the indicator function on $[0,2\pi)$, leading

$$\begin{aligned}\mathcal{L}_c(\boldsymbol{\varepsilon};\boldsymbol{x},\boldsymbol{\theta})&\propto\frac{1}{\sigma^{2N}}e^{-\frac{1}{\sigma^2}\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})-A\tilde{\boldsymbol{I}}\right)^H\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})-A\tilde{\boldsymbol{I}}\right)}\\&\propto\frac{e^{-\frac{1}{\sigma^2}(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))^H(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))-\frac{A^2N}{\sigma^2}+\frac{2A}{\sigma^2}\operatorname{Re}\left\{\tilde{\boldsymbol{I}}^H(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta}))\right\}}}{\sigma^{2N}}\end{aligned} \quad (9)$$

using $\tilde{\boldsymbol{I}}^H\tilde{\boldsymbol{I}}=N$ and assuming $\theta_i\in[0,2\pi)$ to avoid the indicators.

B. E-step: conditional distribution

At the $t+1$ iteration, the E step of the EM algorithm consists in an approximation of the loglikelihood (which is to be maximized) around a previous value of the parameters, namely $\boldsymbol{\varepsilon}^{(t)}$. This approximation is given by

$$Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)})=E_{\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\log\mathcal{L}_c(\boldsymbol{\varepsilon};\boldsymbol{x},\boldsymbol{\theta})\right]. \quad (10)$$

This expected value considers the distribution of $\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}$, given by

$$\begin{aligned}p(\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{\varepsilon})&\propto p(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{\varepsilon})p(\boldsymbol{\theta})\\&\propto\prod_{i=1}^{N}e^{\frac{2A}{\sigma^2}|x_i-\alpha\mu_i(\boldsymbol{\eta})|\cos\left[\theta_i-\arg\left(x_i-\alpha\mu_i(\boldsymbol{\eta})\right)\right]}\end{aligned}$$

where we used the identity

$$\begin{aligned}&\operatorname{Re}\left\{\left(x_i-\alpha\mu_i(\boldsymbol{\eta})\right)e^{-\theta_i}\right\}\\&=|x_i-\alpha\mu_i(\boldsymbol{\eta})|\cos\left(\theta_i-\arg\left(x_i-\alpha\mu_i(\boldsymbol{\eta})\right)\right)\end{aligned} \quad (11)$$

and where

- x_i is the i-th component of the vector $\boldsymbol{x}$
- $\mu_i(\boldsymbol{\eta})$ is the i-th component of the vector $\boldsymbol{\mu}(\boldsymbol{\eta})$
- $\arg(.):\mathbb{C}\rightarrow[0,2\pi)$ is the argument of a complex number.

We deduce the $\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}$ follow independent von Mises distribution with mean $\gamma_i^{(t)}$ and spread parameter $\kappa_i^{(t)}$ defined by

$$\gamma_i^{(t)} = \arg\left(x_i - \alpha^{(t)}\mu_i\left(\boldsymbol{\eta}^{(t)}\right)\right) \tag{12}$$

$$\kappa_i^{(t)} = \frac{2A^{(t)}}{(\sigma^{(t)})^2}\left|x_i - \alpha^{(t)}\mu_i\left(\boldsymbol{\eta}^{(t)}\right)\right|. \tag{13}$$

Indeed, the von Mises probability density function with mean γ and spread parameter κ is expressed as [16, Chap. 3.5]

$$g(\theta;\gamma,\kappa) = \frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\gamma)} \tag{14}$$

where I_p is the modified Bessel function of the first kind and order p. Going back to (10) and using (9)

$$Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}) = K' - N\log\sigma^2 - \frac{1}{\sigma^2}\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})\right)^H\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})\right)$$
$$-\frac{1}{\sigma^2}A^2N + \sum_{i=1}^{N}\frac{2A\left|x_i-\alpha\mu_i(\boldsymbol{\eta})\right|}{\sigma^2}E_{\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i\right)\right].$$

where $\gamma_i = \arg\left(x_i - \alpha\mu_i\left(\boldsymbol{\eta}\right)\right)$.

C. E-step: expectation

We have

$$\begin{aligned}&E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i\right)\right]\\ =&E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i^{(t)}+\gamma_i^{(t)}-\gamma_i\right)\right]\\ =&E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i^{(t)}\right)\right]\cos\left(\gamma_i^{(t)}-\gamma_i\right)\\ &-E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\sin\left(\theta_i-\gamma_i^{(t)}\right)\right]\sin\left(\gamma_i^{(t)}-\gamma_i\right).\end{aligned}$$

Yet, $\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}$ has von Mises distribution with parameters $\gamma_i^{(t)}$ and $\kappa_i^{(t)}$. Eqs. (3.5.25) and (3.5.26) in [16] lead

$$E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i^{(t)}\right)\right] = \frac{I_1\left(\kappa_i^{(t)}\right)}{I_0\left(\kappa_i^{(t)}\right)},$$

$$E_{\theta_i|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\sin\left(\theta_i-\gamma_i^{(t)}\right)\right] = 0.$$

So

$$E_{\boldsymbol{\theta}|\boldsymbol{x},\boldsymbol{\varepsilon}^{(t)}}\left[\cos\left(\theta_i-\gamma_i\right)\right] = \frac{I_1\left(\kappa_i^{(t)}\right)}{I_0\left(\kappa_i^{(t)}\right)}\cos\left(\gamma_i^{(t)}-\gamma_i\right) \tag{15}$$

and therefore

$$\begin{aligned}Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}) =& K' - N\log\sigma^2 - \frac{\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})\right)^H\left(\boldsymbol{x}-\alpha\boldsymbol{\mu}(\boldsymbol{\eta})\right)}{\sigma^2} - \frac{A^2N}{\sigma^2}\\ &+\sum_{i=1}^{N}\frac{2A}{\sigma^2}\left|x_i-\alpha\mu_i(\boldsymbol{\eta})\right|\frac{I_1\left(\kappa_i^{(t)}\right)}{I_0\left(\kappa_i^{(t)}\right)}\cos\left(\gamma_i^{(t)}-\arg\left(x_i-\alpha\mu_i\left(\boldsymbol{\eta}\right)\right)\right).\end{aligned} \tag{16}$$

Finally, defining

$$w_i^{(t)} = \frac{I_1\left(\kappa_i^{(t)}\right)}{I_0\left(\kappa_i^{(t)}\right)} \tag{17}$$

$$\boldsymbol{a}_t = \left[w_1^{(t)}e^{j\gamma_1^{(t)}} \quad \ldots \quad w_N^{(t)}e^{j\gamma_N^{(t)}}\right]^T \tag{18}$$

and using (11) with $\gamma_i^{(t)}$ instead of θ_i, the objective function (16) can be rewritten as

$$\begin{aligned}Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}) =& \frac{A^2}{\sigma^2}\boldsymbol{a}_t^H\boldsymbol{a}_t - \frac{A^2}{\sigma^2}N + K' - N\log\sigma^2\\ &-\frac{\left(\boldsymbol{x}-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})-A\boldsymbol{a}_t\right)^H\left(\boldsymbol{x}-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})-A\boldsymbol{a}_t\right)}{\sigma^2}.\end{aligned} \tag{19}$$

D. M-step: optimisation with respect to σ^2

The M step of the EM algorithm consists in updating the value of the vector of parameters $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon}^{(t+1)} = \arg\max_{\boldsymbol{\varepsilon}} Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}). \tag{20}$$

Differentiating (19) w.r.t. σ^2 yields the update equation

$$\begin{aligned}\left(\sigma^2\right)^{(t+1)} =& \frac{1}{N}\left\|\boldsymbol{x}-\rho^{(t+1)}e^{j\varphi^{(t+1)}}\boldsymbol{\mu}(\boldsymbol{\eta}^{(t+1)})-A^{(t+1)}\boldsymbol{a}_t\right\|^2\\ &+(A^{(t+1)})^2\left(1-\frac{1}{N}\sum_{i=1}^{n}\left(w_i^{(t)}\right)^2\right)\end{aligned} \tag{21}$$

where we used the notation $\|\boldsymbol{z}\|^2 = \boldsymbol{z}^H\boldsymbol{z}$ for any $\boldsymbol{z}\in\mathbb{C}$.

E. M-step: optimisation with respect to $\rho,\varphi,\boldsymbol{\eta}$

The objective function (19) can be simplified as

$$Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}) = -\frac{1}{\sigma^2}\left\|\boldsymbol{x}-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})-A\boldsymbol{a}_t\right\|^2 + K''$$

where K'' is independent of ρ,φ and $\boldsymbol{\eta}$. We define

$$\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})} = \boldsymbol{\mu}\left(\boldsymbol{\eta}\right)\left(\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)\right)^{-1}\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H \tag{22}$$

$$\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}^{\perp} = I_N - \boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})} \tag{23}$$

the projection matrices on the space spanned by $\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}$ and on its orthogonal, respectively. We have

$$\begin{aligned}&\left\|\boldsymbol{x}-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})-A\boldsymbol{a}_t\right\|^2\\ =&\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})\right\|^2 + \left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}^{\perp}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)\right\|^2.\end{aligned} \tag{24}$$

On one hand, maximizing $Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)})$ w.r.t. ρ,φ is equivalent to minimizing $\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})\right\|^2$. But

$$\begin{aligned}&\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})\right\|^2\\ =&\left\|\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)\left[\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)}{\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)}-\rho e^{j\varphi}\right]\right\|^2\end{aligned}$$

and this quantity is equal to 0 (hence minimal) for

$$\rho e^{j\varphi} = \frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)}{\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}\right)}$$

yielding to the update equations

$$\rho^{(t+1)} = \left|\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\left(\boldsymbol{x}-A^{(t+1)}\boldsymbol{a}_t\right)}{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)}\right| \tag{25}$$

$$\varphi^{(t+1)} = \arg\left(\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\left(\boldsymbol{x}-A^{(t+1)}\boldsymbol{a}_t\right)}{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)}\right). \tag{26}$$

On the other hand, going back to (24), at the optimal we have (using Pythagora's theorem)

$$\begin{aligned}\left\|\boldsymbol{x}-\rho e^{j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})-A\boldsymbol{a}_t\right\|^2 &= \left\|\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)\right\|^2 \\ &= \|\boldsymbol{x}-A\boldsymbol{a}_t\|^2-\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A\boldsymbol{a}_t\right)\right\|^2.\end{aligned}$$

Then, maximizing $Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)})$ w.r.t. $\boldsymbol{\eta}$ is equivalent to maximize

$$\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}(\boldsymbol{x}-A\boldsymbol{a}_t)\right\|^2$$

yielding to the update equation

$$\boldsymbol{\eta}^{(t+1)}=\arg\max_{\boldsymbol{\eta}}\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-A^{(t+1)}\boldsymbol{a}_t\right)\right\|^2. \quad (27)$$

F. M-step: optimisation with respect to A

The objective function (19) can be expressed as

$$\begin{aligned}Q(\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}^{(t)}) =& \frac{A}{\sigma^2}\boldsymbol{x}^H\boldsymbol{a}_t+\frac{A}{\sigma^2}\boldsymbol{a}_t^H\boldsymbol{x}-\frac{A}{\sigma^2}\rho e^{-j\varphi}\boldsymbol{\mu}(\boldsymbol{\eta})^H\boldsymbol{a}_t \\ &-\frac{A}{\sigma^2}\rho e^{j\varphi}\boldsymbol{a}_t^H\boldsymbol{\mu}(\boldsymbol{\eta})-\frac{A^2}{\sigma^2}N+K''' \quad (28)\end{aligned}$$

where K''' is independent of A. Differentiating this expression w.r.t. A yields the update equation

$$A^{(t+1)}=\frac{1}{N}\mathrm{Re}\left\{\boldsymbol{a}_t^H\left(\boldsymbol{x}-\rho^{(t+1)}e^{j\varphi^{(t+1)}}\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)\right)\right\}. \quad (29)$$

We can note that

$$\rho^{(t+1)}e^{j\varphi^{(t+1)}}=\left(\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\left(\boldsymbol{x}-A^{(t+1)}\boldsymbol{a}_t\right)}{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)}\right) \quad (30)$$

can be injected in (29) to give the update equation

$$A^{(t+1)}=\frac{\mathrm{Re}\left\{\boldsymbol{a}_t^H\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)}\boldsymbol{x}\right\}}{N-\mathrm{Re}\left\{\boldsymbol{a}_t^H\boldsymbol{\Pi}_{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(t+1)}\right)}\boldsymbol{a}_t\right\}}. \quad (31)$$

Note that this equation can be injected into (27) to give

$$\boldsymbol{\eta}^{(t+1)}=\arg\max_{\boldsymbol{\eta}}\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\left(\boldsymbol{x}-\frac{\mathrm{Re}\left\{\boldsymbol{a}_t^H\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\boldsymbol{x}\right\}}{N-\mathrm{Re}\left\{\boldsymbol{a}_t^H\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\boldsymbol{a}_t\right\}}\boldsymbol{a}_t\right)\right\|^2. \quad (32)$$

G. Algorithm:

For $t=0$, i.e., for initialization, we might compute $\left\{\left(\boldsymbol{\eta}^{(0)}\right)^T,\rho^{(0)},\phi^{(0)},\left(\sigma^2\right)^{(0)}\right\}$ thanks to the standard MLE [11]. In other words:

$$\boldsymbol{\eta}^{(0)}=\arg\max_{\boldsymbol{\eta}}\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\boldsymbol{x}\right\|^2 \quad (33)$$

$$\rho^{(0)}=\left|\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)^H\boldsymbol{x}}{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)}\right| \quad (34)$$

$$\varphi^{(0)}=\arg\left\{\frac{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)^H\boldsymbol{x}}{\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)^H\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)}\right\} \quad (35)$$

$$(\sigma^2)^{(0)}=\frac{1}{N}\left\|\boldsymbol{x}-\rho^{(0)}e^{j\varphi^{(0)}}\boldsymbol{\mu}\left(\boldsymbol{\eta}^{(0)}\right)\right\|^2. \quad (36)$$

Once the interference is detected, due to the fact that the interference is much more powerful than the GNSS signal masked in the noise, we keep the MLE estimators for $\boldsymbol{\eta}^{(1)},\rho^{(1)}\varphi^{(1)}$ and $(\sigma^2)^{(1)}$, and we initialize the interference power as

$$A^{(1)}=\frac{1}{N}\sqrt{\boldsymbol{x}^H\boldsymbol{x}}. \quad (37)$$

Then, we compute the von Mises parameters, $\mu_i^{(t)}$ and $\kappa_i^{(t)}$, and trigonometric moments $w_i^{(t)}$, using Eqs. (12), (13) and (17) respectively. Finally, we update the parameters of interest in the order $\boldsymbol{\eta},A,\rho,\varphi$ and σ^2 with Eqs. (27),(31),(25),(26) and (21) respectively. Note that Eqs. (32) and (33) don't have closed-form expressions. However, a grid search approach can be used to maximize the corresponding functions. Also, a little simplification can be made to optimize (27), using (32) where $A^{(t+1)}$ is replaced by $A^{(t)}$ to update $\boldsymbol{\eta}$. With this approach, the same function can be used to solve both (33) and (32).

IV. Experiments

Let us consider the case where a GPS L1 C/A signal [1] is attacked by a jammer that is generating a linear frequency modulation (LFM) signal [17], which is defined as

$$I(t)=\Pi_T(t)\times e^{j\pi\beta_c t^2+j\phi},\ \Pi_T(t)=\begin{cases}A_i & \text{for } 0\le t<T\\ 0 & \text{otherwise}\end{cases} \quad (38)$$

where β_c is the chirp rate and A_i is the amplitude. For this particular scenario, we set the waveform period as $T=N\cdot T_s$, i.e. equal to the integration time. The instantaneous frequency is $f(t)=\frac{1}{2\pi}\frac{d}{dt}\left(\pi\beta_c t^2\right)=\beta_c t$, and therefore the waveform bandwidth is $B_c=\beta_c T$. We consider the case where, after the Hilbert filter, the chirp is located at the baseband frequency, i.e., the central frequency of the chirp is $f_i=0$. Then, the waveform can be rewritten as,

$$I(t)=\Pi_T(t)\times e^{j\pi\beta_c(t-T/2)^2+j\phi}. \quad (39)$$

The mean squared error (MSE) for the parameters of interest $\boldsymbol{\eta}^T$ are shown in Figures 1-2, w.r.t. the SNR at the output of the matched filter (i.e., SNR_{OUT}) and considering the following setup: a GNSS receiver with sampling frequency $F_s=4$ MHz, and a chirp bandwidth $B_c=2$ MHz, with initial phase $\phi=0$, amplitude $A_i=40$ and integration times $T=\{1,2\}$ ms. The EM algorithm integrates a stopping criterion that compares each iteration the variation of the noise variance. The maximum number of iterations is set to 20 and the number of Monte Carlo is set to 1000 runs. In the results one can observe i) the $\sqrt{CRB}$ (refer to [11]), which represents the asymptotic estimation performance of the parameters without any source of interference, ii) the $\sqrt{MCRB+Bias^2}$ which represents the asymptotic estimation performance of the parameters with an interference source (refer to [12], [18]) and describes the MSE of the misspecified maximum likelihood estimator [19], and iii) the Root MSE ($\sqrt{MSE}$) of the proposed EM algorithm. Note that the EM algorithm is not biased and it is able to correct the effects generated by the interference. Moreover, we can also verify that the error is almost the same as that obtained for the MLE under the case without interference. Such results validate and prove the good performance of the proposed algorithm.

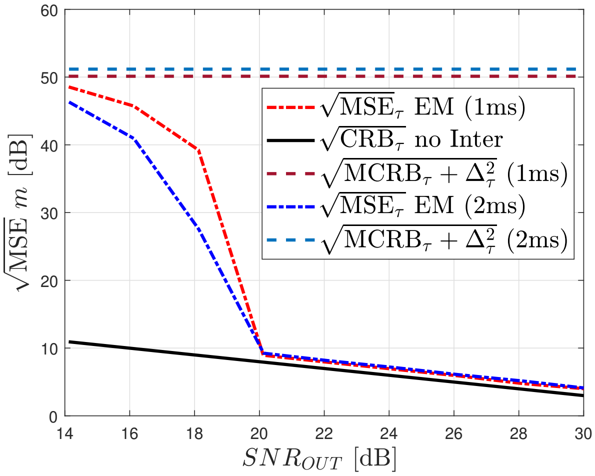

Fig. 1. EM algorithm estimator RMSE for time-delay of the GPS L1 C/A signal received along with a centered chirp signal of bandwidth $B_c = 2$ MHz and amplitude $A = 40$. The sampling frequency is set to $F_s = 4$ MHz.

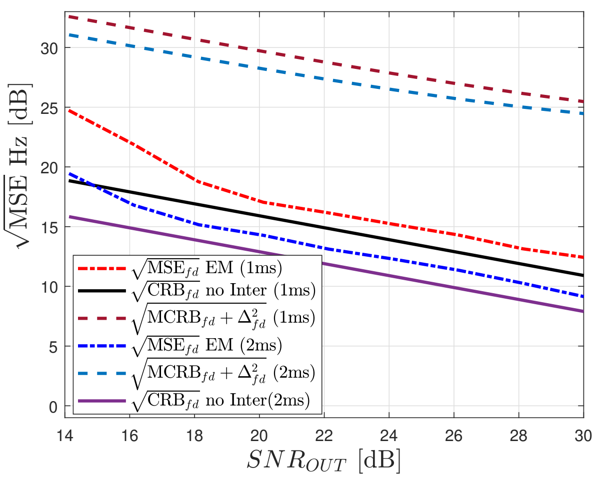

Fig. 2. EM algorithm estimator RMSE for Doppler of the GPS L1 C/A signal received along with a centered chirp signal of bandwidth $B_c = 2$ MHz and amplitude $A = 40$. The sampling frequency is set to $F_s = 4$ MHz.

V. Conclusion

It is well-documented that interferences may have a huge impact of the GNSS receivers' performance. In this article, we propose an EM algorithm that consider the structure of one of the most notable family of interferences, i.e., CE, in order to estimate jointly the parameters of interest along with the CE of the interference signal. We illustrate the MSE of the time-delay and Doppler estimation performances of the proposed EM algorithm considering a chirp interference jamming and a GPS L1 C/A signal. Results were provided to show the good performance of the proposed algorithm.

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