Sparse Estimation of Multipath Biases for Global Navigation Satellite Systems

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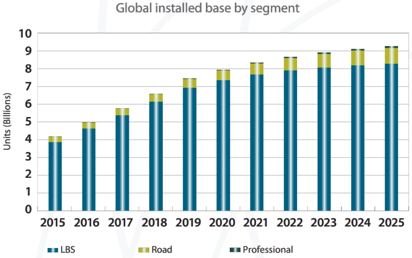
Outline

Introduction

GPS Applications[1]

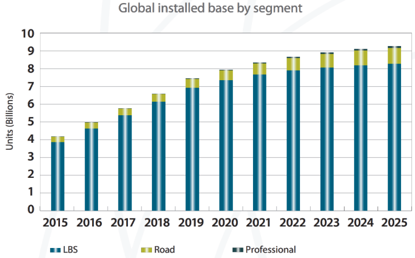


LBS: Location-Based Services

80% of Smartphones

[1] European GNSS Agency. *GNSS Market report*. Issue 5. 2017.

GPS Applications[1]



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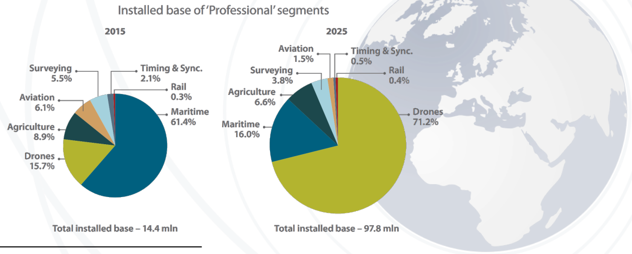


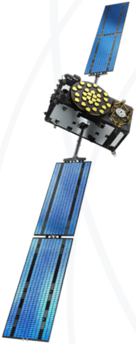
[1] European GNSS Agency. *GNSS Market report*. Issue 5. 2017.

GNSS

Global Navigation Satellite Systems

- GPS: USA,1973
- GLONASS: URSS, 1976
- Compass-Beidou: China, 1983 (Beidou) 2007 (Compass)
- Galileo: EU, 1999
- QZSS: Japan, 2002
- IRNSS: India, 2006

GNSS Satellites



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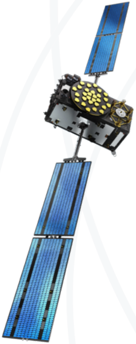

$\sim$ 30 satellites/constellation

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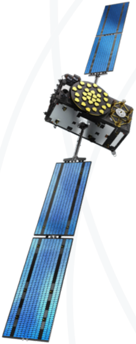
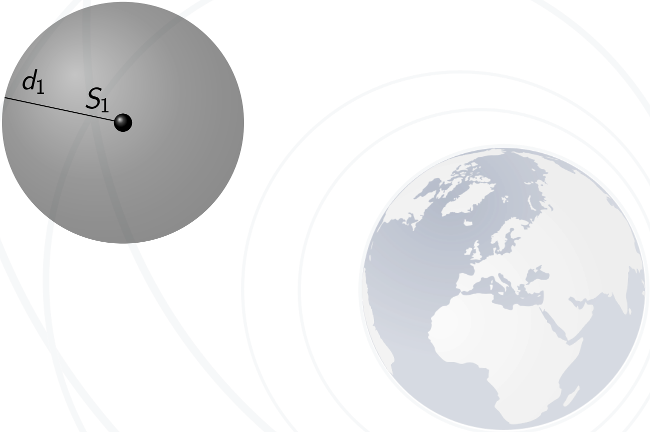


$\sim$ 30 satellites/constellation

$\sim$ 20000 km altitude

Principle: trilateration



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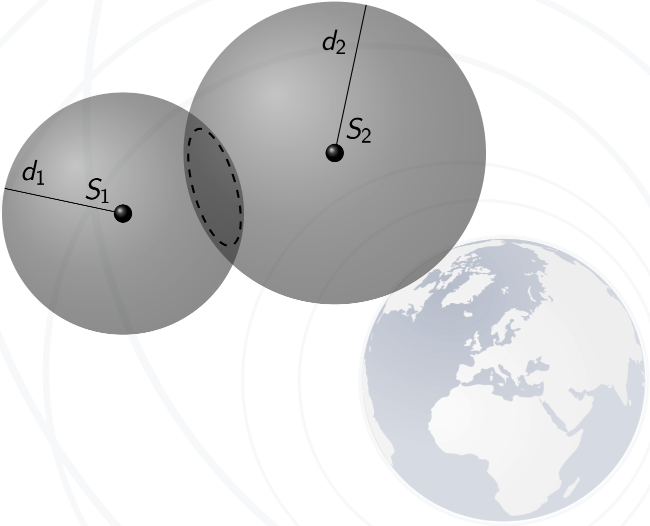

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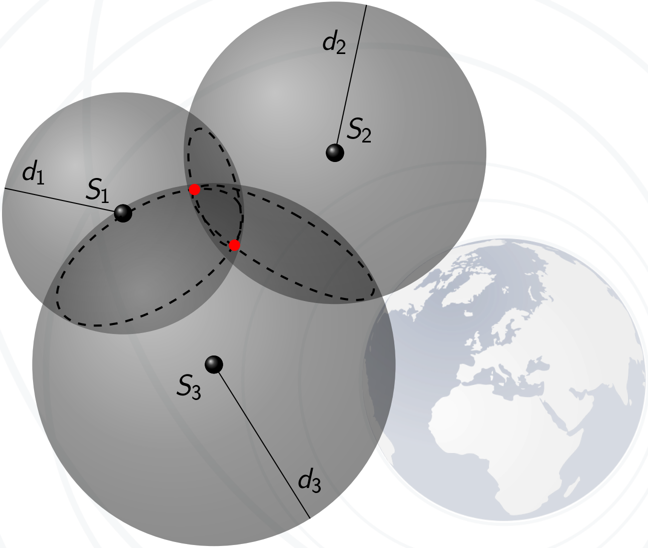
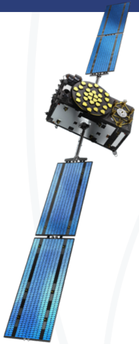


Signal propagation (1)



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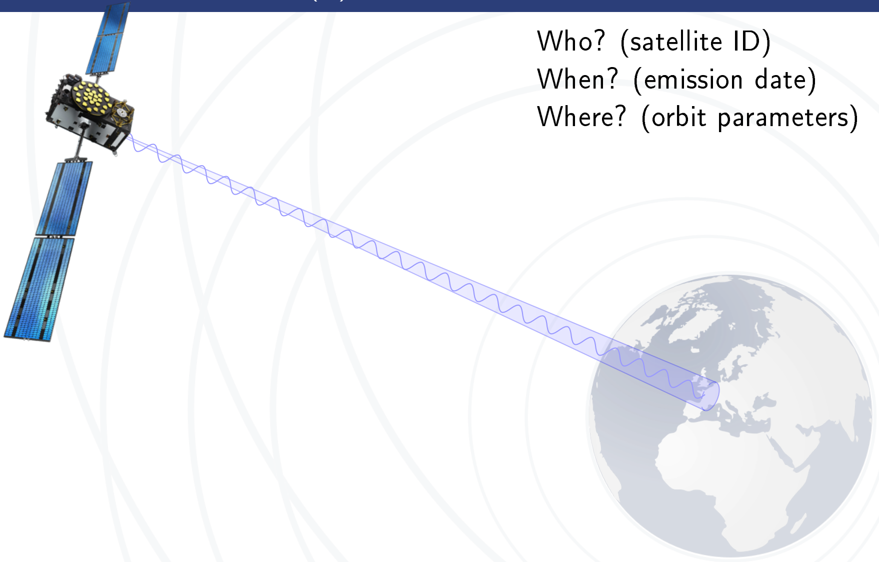

Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)

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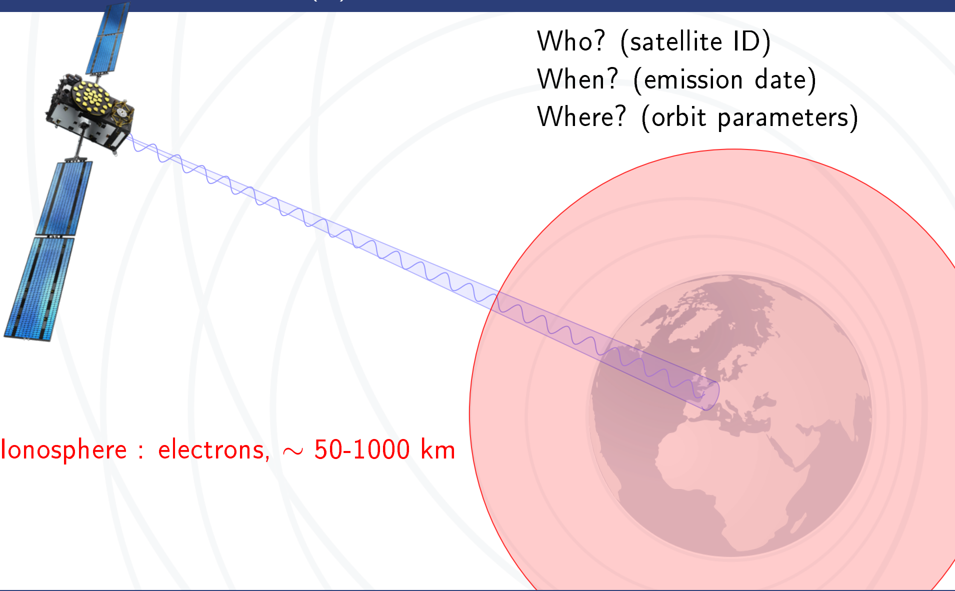

Ionosphere : electrons, ∼ 50-1000 km

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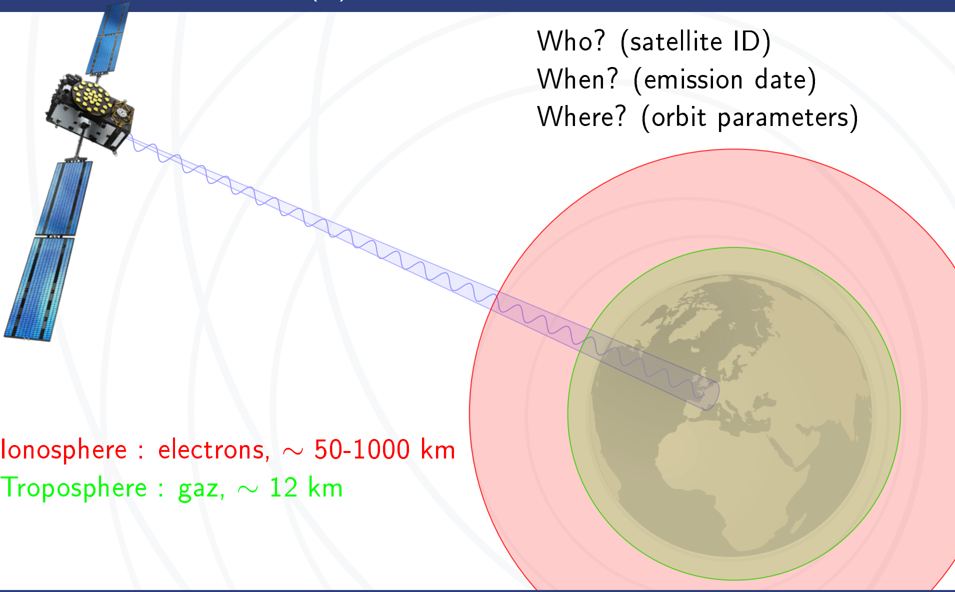

Ionosphere : electrons, ∼ 50-1000 km
Troposphere : gaz, ∼ 12 km

Signal propagation (2)

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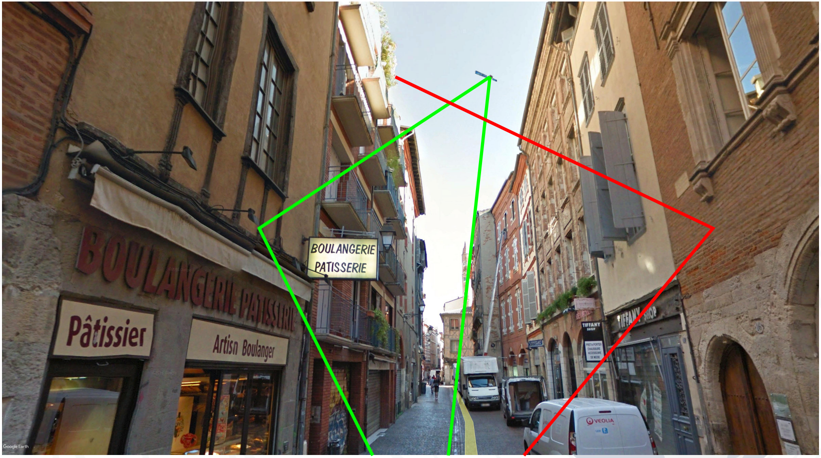
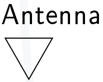


GNSS receiver



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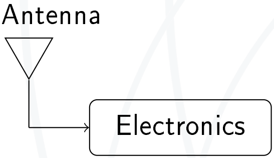

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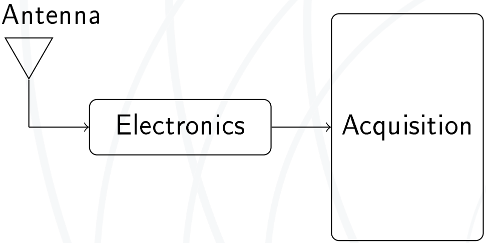

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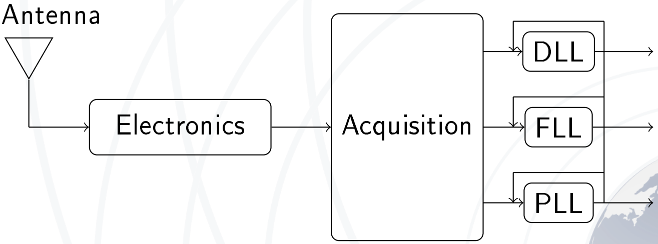

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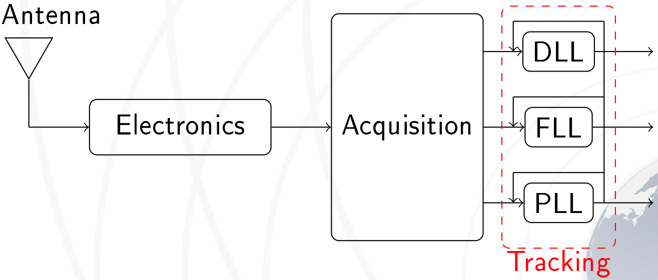

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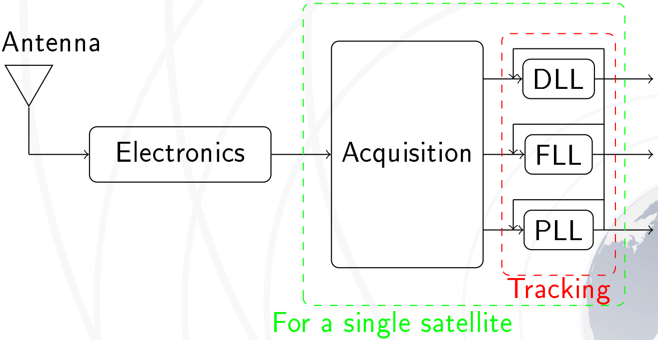

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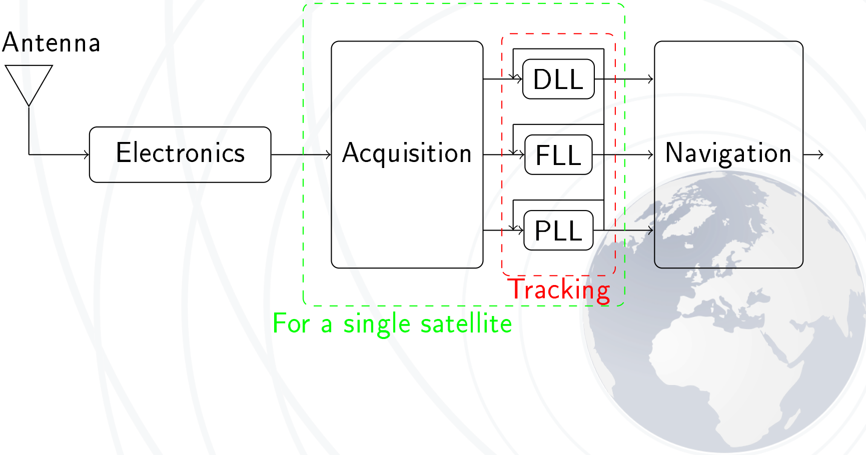

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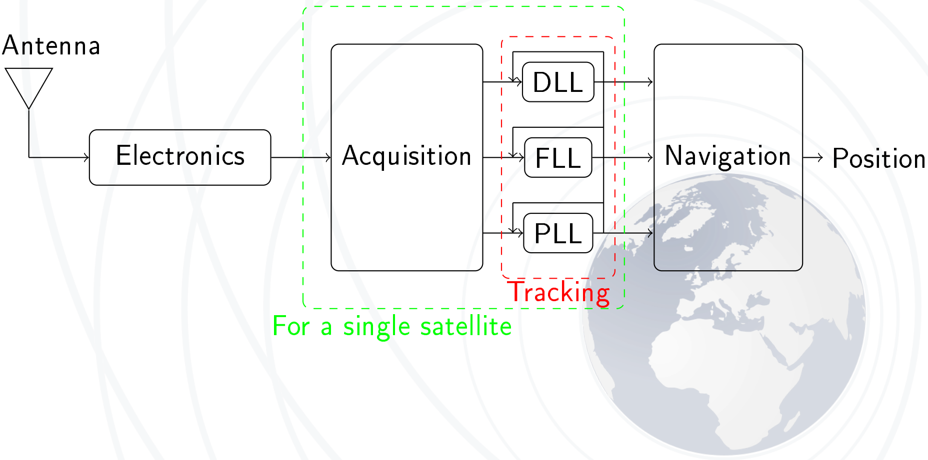

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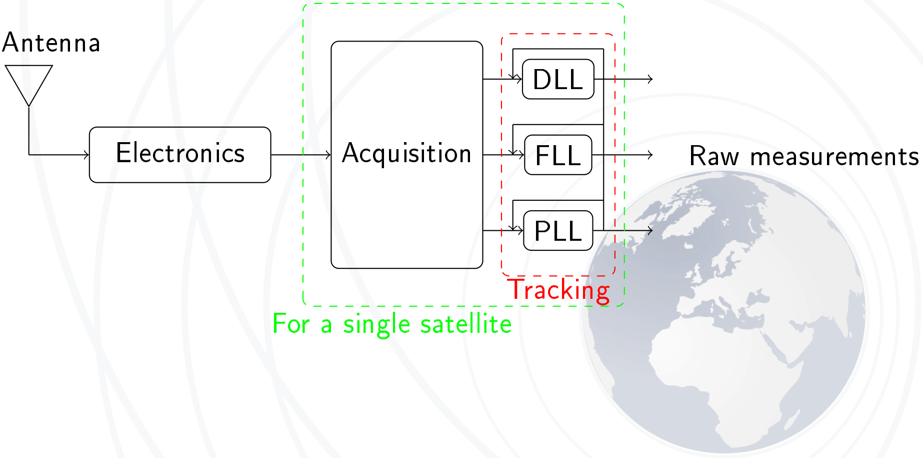
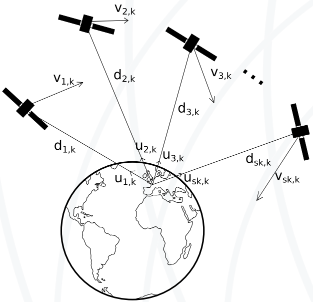


State Space Model

Navigation problem[2]



Measurements for satellite i at time k

$$\rho_{i,k} = \underbrace{\|\boldsymbol{r_k} - \boldsymbol{r}_{i,k}\|_2}_{d_{i,k}} + b_k + \varepsilon_{i,k}$$

$$\dot{\rho}_{i,k} = (\boldsymbol{v}_k - \boldsymbol{v}_{i,k})^T \boldsymbol{u}_{i,k} + \dot{b}_k + e_{i,k}$$

- $\boldsymbol{r}_k$: receiver's position
- $\boldsymbol{r}_{i,k}$: satellite's position
- $\boldsymbol{v}_k$: receiver's velocity
- $\boldsymbol{v}_{i,k}$: satellite's velocity
- b_k: receiver's clock bias
- $\dot{b}_k$: receiver's clock drift
- $\varepsilon_{i,k}$: pseudorange error
- $e_{i,k}$: pseudospeed error

State vector:

$$\boldsymbol{\xi}_k = \{\boldsymbol{r}_k, \boldsymbol{v}_k, b_k, \dot{b}_k\} \in \mathbb{R}^8$$

[2] Paul D. Groves. *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems.* 1st ed. Artech House Publishers, 2008.

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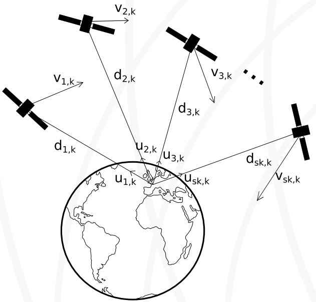

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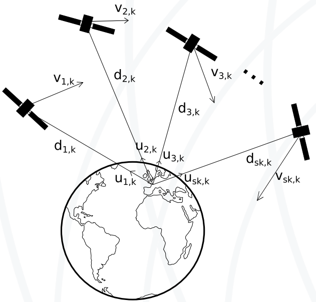

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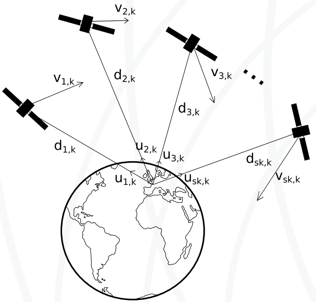

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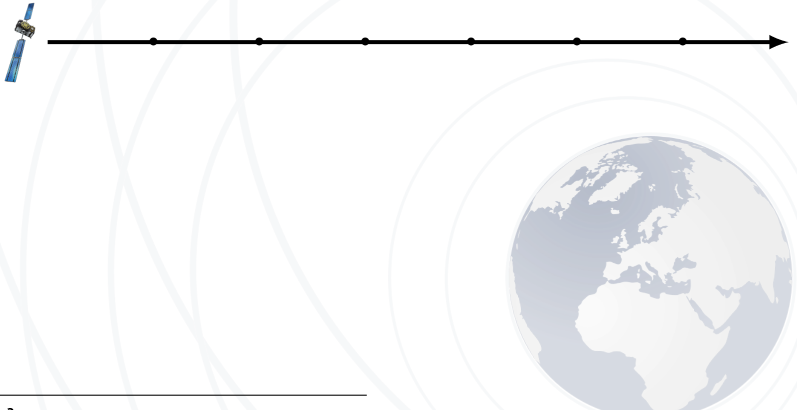
GNSS Error Budget[3]



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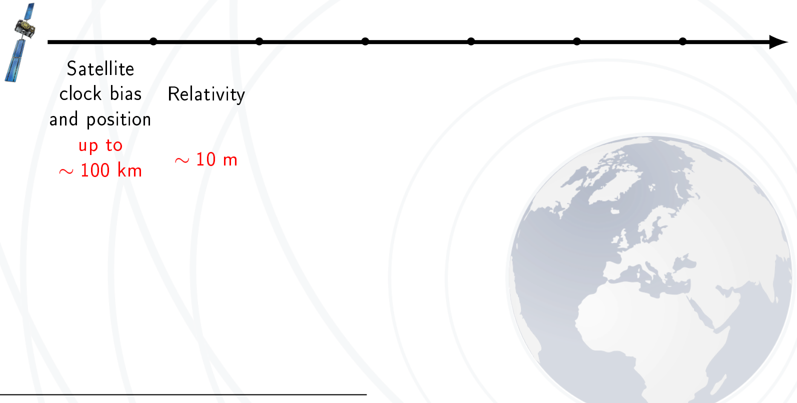

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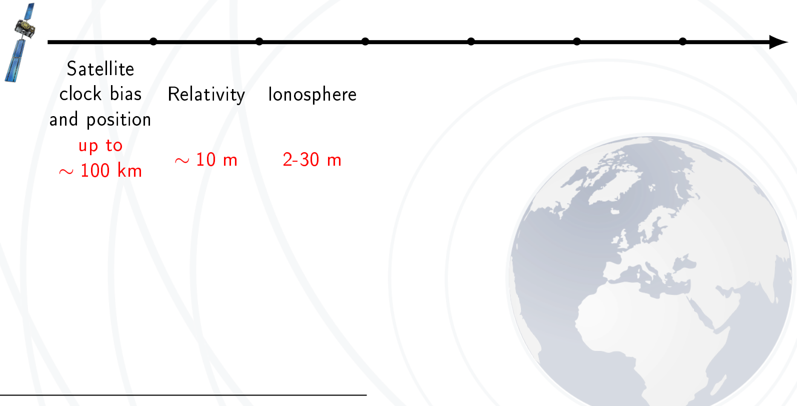

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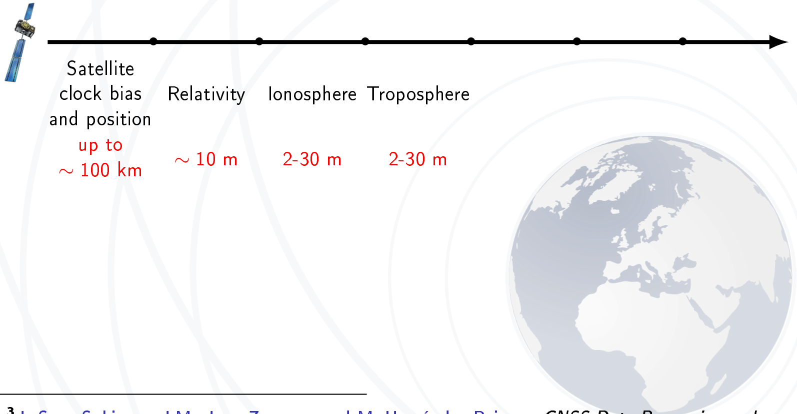

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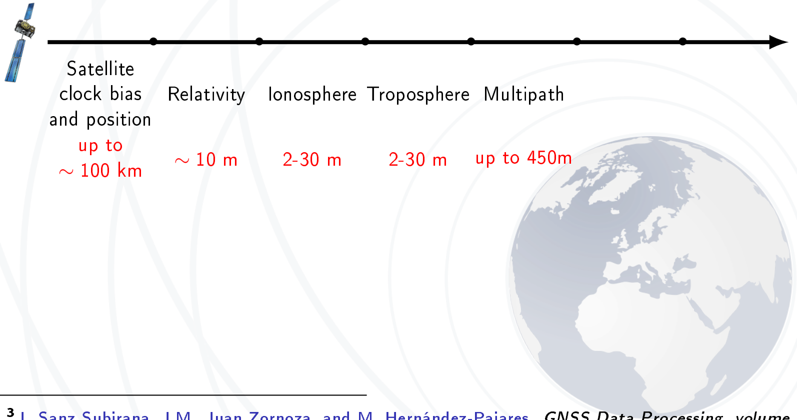

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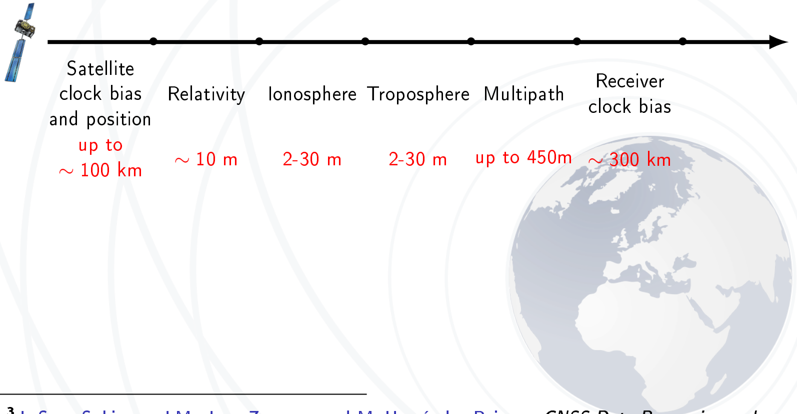

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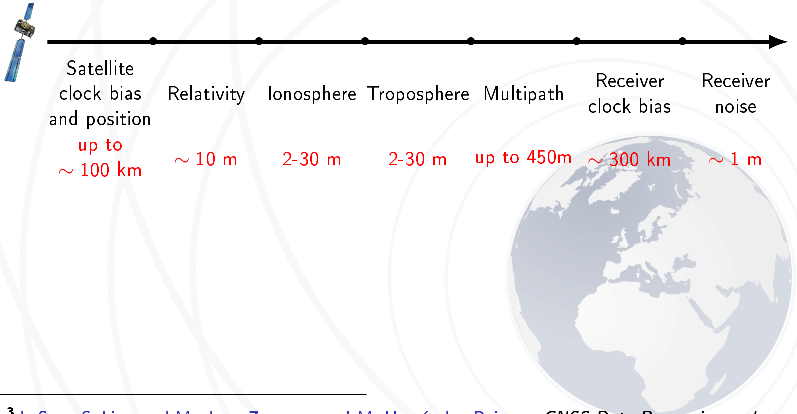

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GNSS Error Budget[3]

Satellite clock bias and position	Relativity	Ionosphere	Troposphere	Multipath	Receiver clock bias	Receiver noise
up to $\sim$ 100 km	$\sim$ 10 m	2-30 m	2-30 m	up to 450m	$\sim$ 300 km	$\sim$ 1 m
Ephemeris or Precise orbits	Explicit	Klobuchar, bi-frequency or SBAS	Niell or SBAS		Estimated	Statistical model

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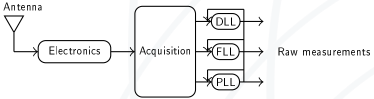
Multipath Mitigation[4]



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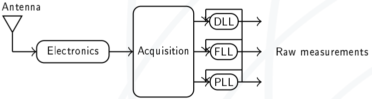

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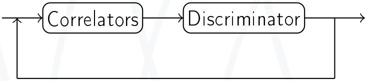

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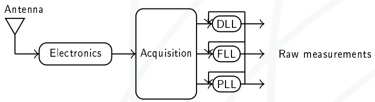

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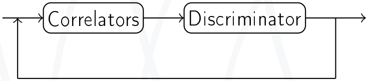

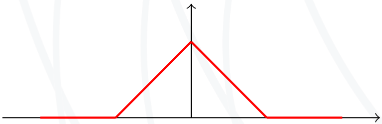

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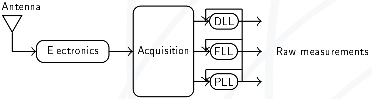

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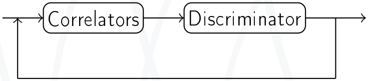

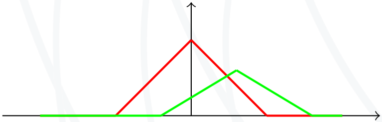

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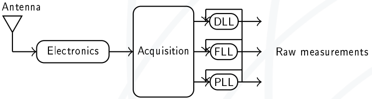

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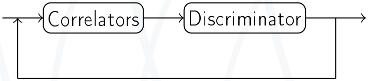

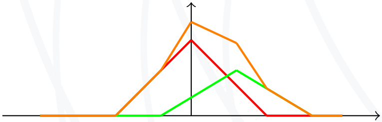

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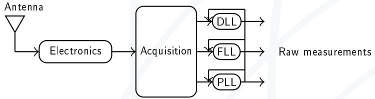

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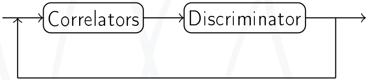

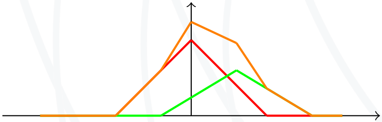

GNSS signals

Code waveform

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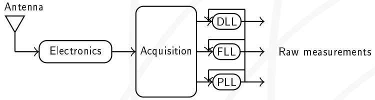

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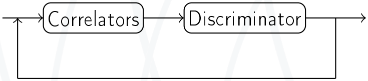

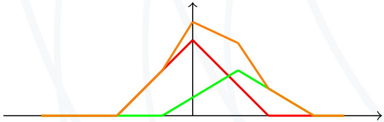

GNSS signals

Code waveform

Antenna

Geometry or spatial processing

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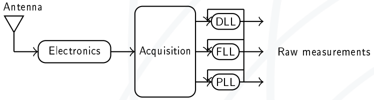
Multipath Mitigation[4]



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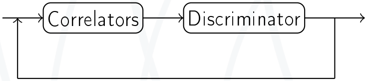

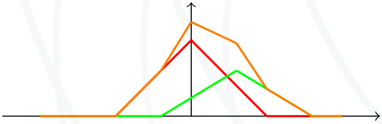

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Digital signal
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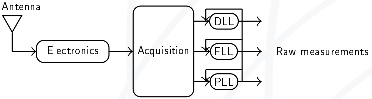

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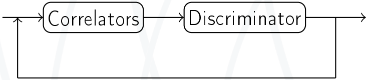

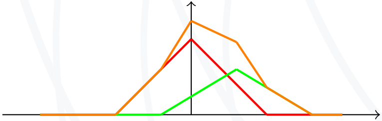

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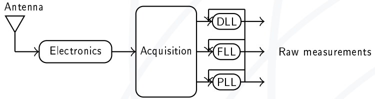

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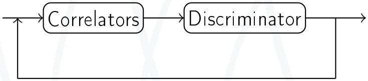

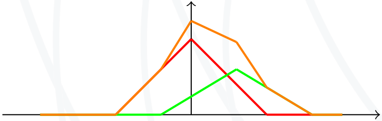

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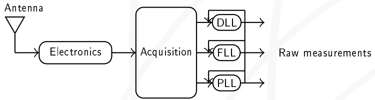

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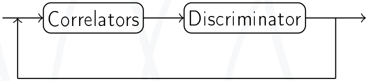

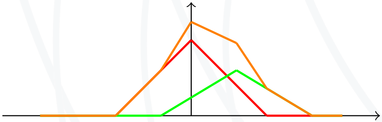

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System equations

Measurements $\boldsymbol{z}_k \in \mathbb{R}^{2s_k}$

Hypothesis: models for everything except multipath and noise[5,6,7]

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\xi_k) + \boldsymbol{m}_k + \boldsymbol{n}_k \quad \text{with} \quad \begin{array}{l} \boldsymbol{h}_k \text{ known and nonlinear} \\ \boldsymbol{m}_k \text{ unknown} \\ \boldsymbol{n}_k \sim \mathcal{N}(\boldsymbol{n}_k; 0, \boldsymbol{R}_k) \end{array}$$

Extended Kalman Filter (EKF)

Filter considering a state propagation model (standard: $\boldsymbol{m}_k = 0$)

Fault Detection and Exclusion (FDE)

Remove faulty satellites based on hypothesis tests on the residuals

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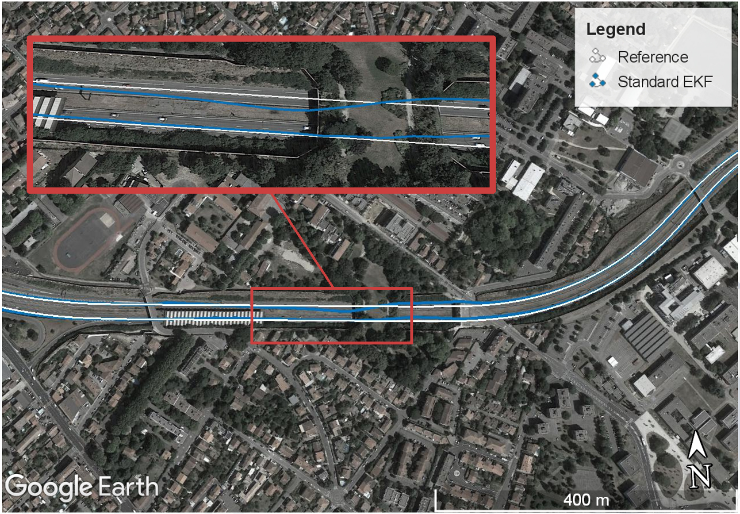

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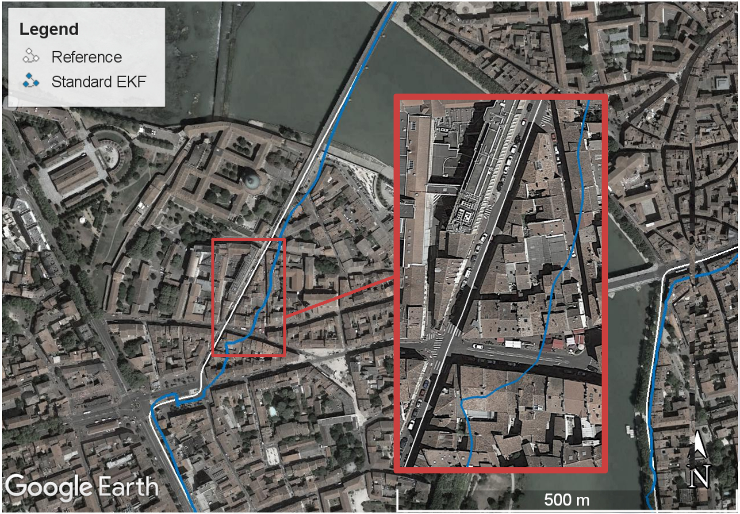

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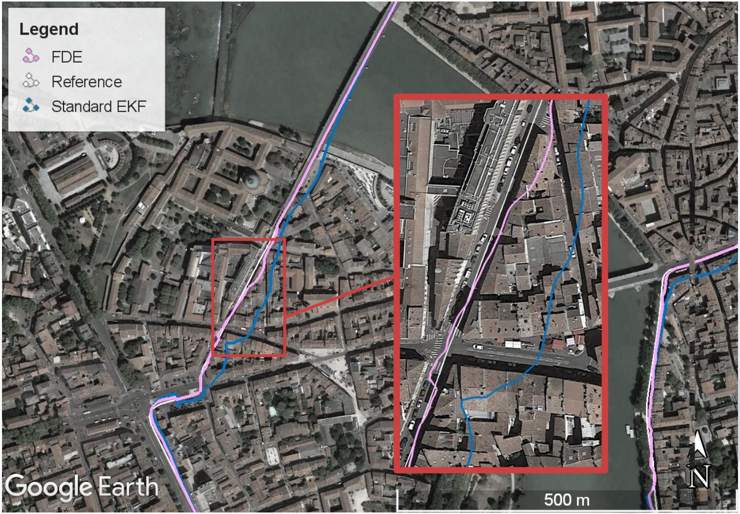

Fault Detection and Exclusion (FDE)

First idea

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{\xi}_k) + \boldsymbol{m}_k + \boldsymbol{n}_k$$

Assumption: 6 satellites $\equiv$ 12 measurements
$\rightarrow$ maybe 4 measurements suffer from MP

$$\boldsymbol{m}_k = \begin{pmatrix} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{pmatrix} \rightarrow \boldsymbol{m}_k = \begin{pmatrix} 0 \\ \square \\ 0 \\ \square \\ 0 \\ \square \\ 0 \\ 0 \\ 0 \\ 0 \\ \square \\ 0 \end{pmatrix}$$

$\rightarrow$ Sparse estimation to estimate MP biases on raw measurements

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Sparse Estimation

Sparse Regularization

Measurements

$$\tilde{\boldsymbol{y}}_k = \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k + \tilde{\boldsymbol{n}}_k \quad \text{with} \quad \tilde{\boldsymbol{H}}_k \text{ low rank}$$

$\Rightarrow$need for appropriate regularization

Assumption: $\boldsymbol{\theta}_k$ is sparse $\rightarrow$ minimize $\|\boldsymbol{\theta}_k\|_0$

LASSO[8]

$$\underset{\boldsymbol{\theta}_k}{\arg\min} \left\{ \underbrace{\frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2}_{\text{data-fidelity term}} + \underbrace{\lambda_k \|\boldsymbol{\theta}_k\|_1}_{\text{regularization term}} \right\}$$

Weighted-ℓ_1[9]

$$\underset{\boldsymbol{\theta}_k}{\arg\min} \left\{ \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k \|\boldsymbol{W}_k\boldsymbol{\theta}_k\|_1 \right\}$$

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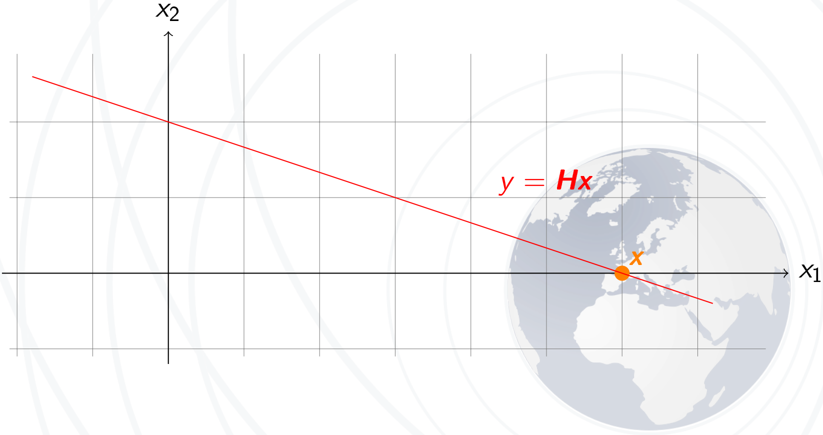

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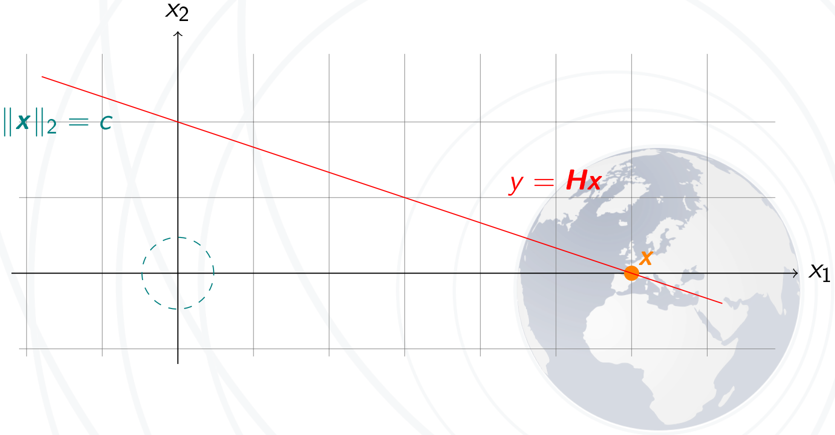

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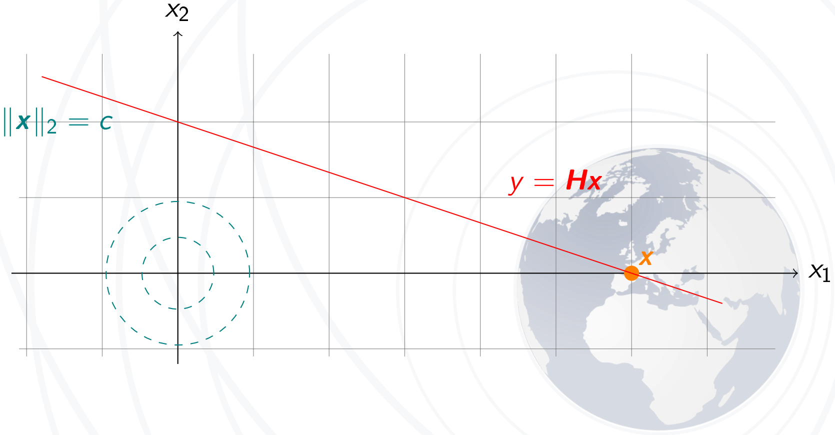

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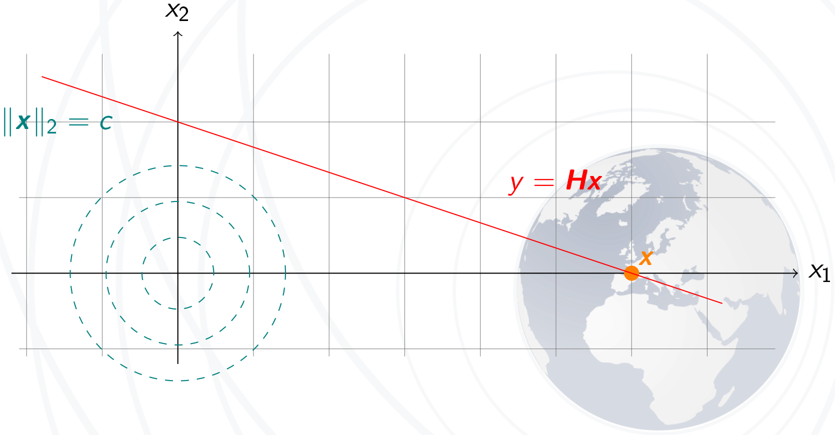

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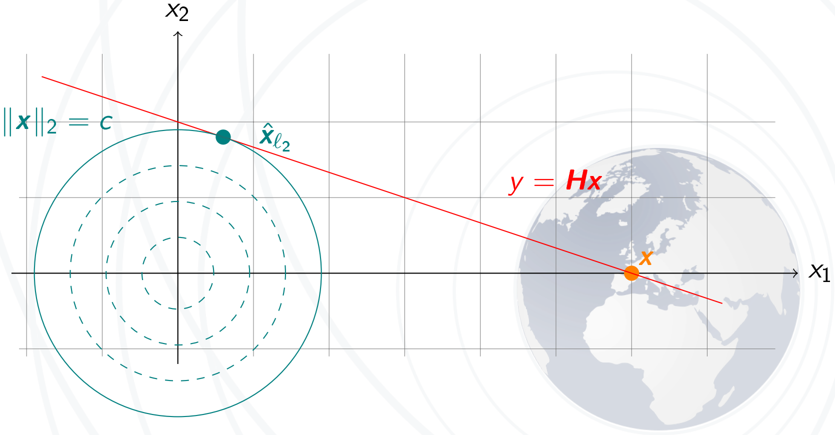

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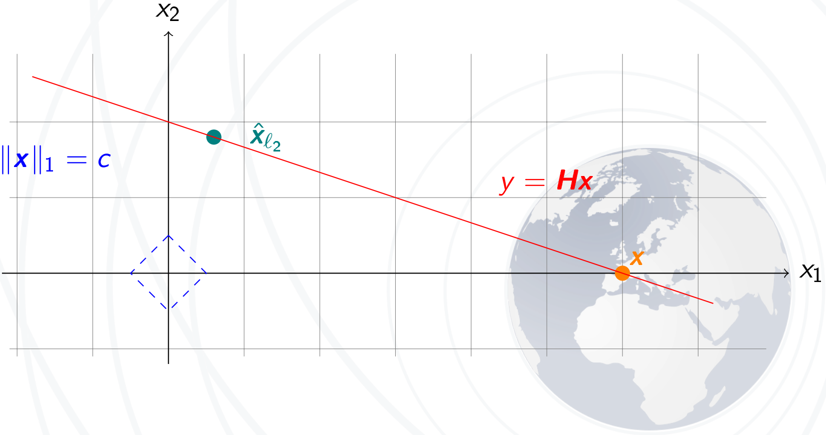

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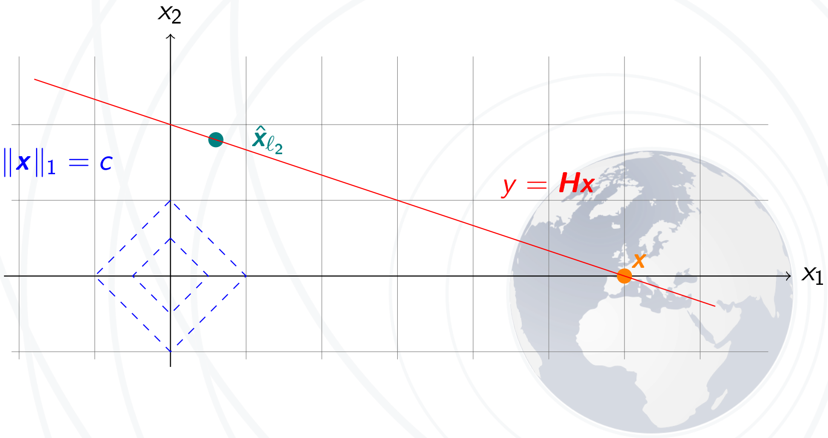

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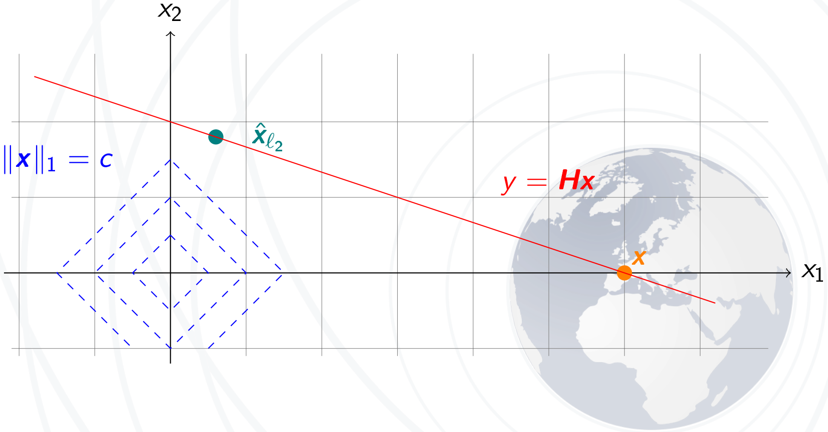

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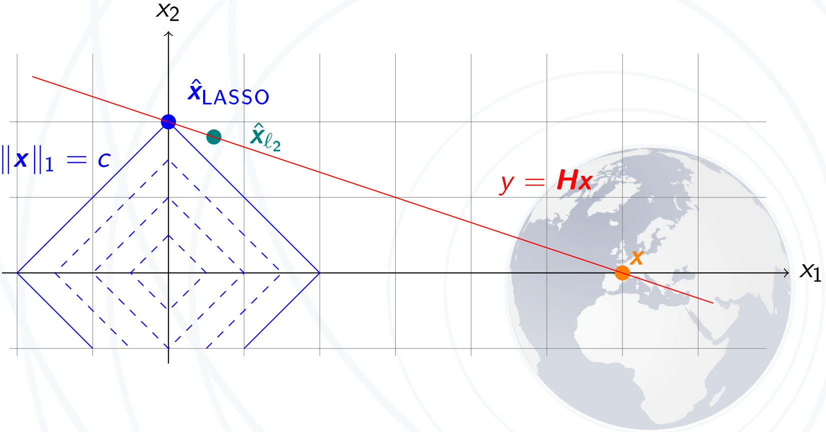
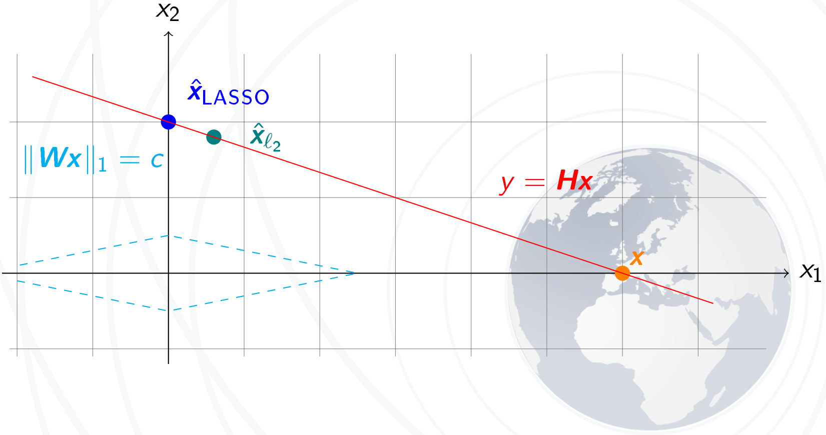


Toy Example

$$\boldsymbol{H} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}$$



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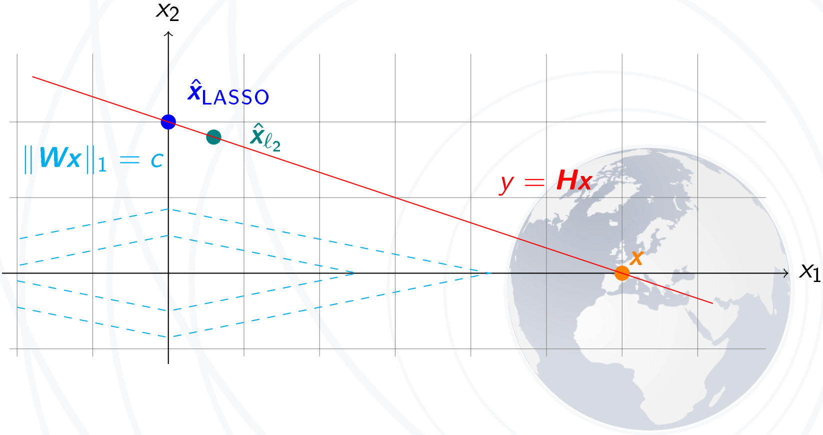

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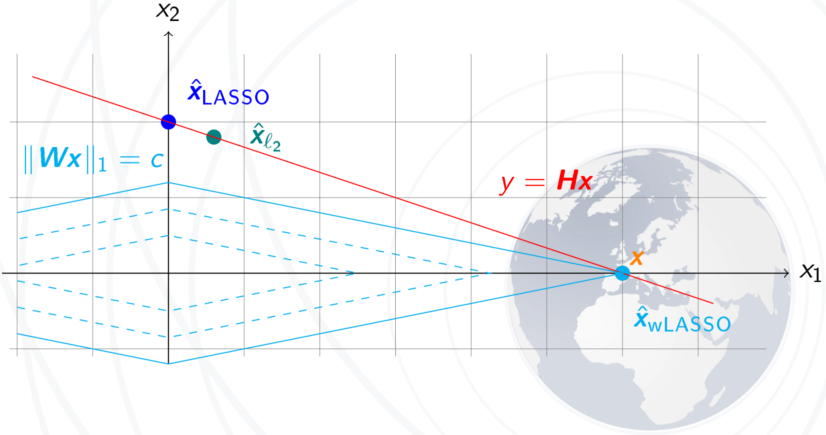

Application to Multipath Bias Estimation[10,11]

Measurements

$$z_k - h_k(\hat{\xi}_{k|k-1}) = H_k(\xi_k - \hat{\xi}_{k|k-1}) + m_k + n_k$$
$$\Leftrightarrow y_k = H_k x_k + m_k + n_k$$

Assumption

$$m_k \text{ is sparse}$$

Weighted-ℓ_1

$$\arg\min_{x_k, m_k} \left\{ \frac{1}{2} \| y_k - H_k x_k - m_k \|_2^2 + \lambda_k \| W_k m_k \|_1 \right\}$$

[10] Julien Lesouple, Thierry Robert, Mohamed Sahmoudi, Jean-Yves Tourneret, and Willy Vigneau. "Multipath Mitigation for GNSS Positioning in Urban Environment Using Sparse Estimation". In: *IEEE Trans. Intell. Transp. Syst.* (2019).

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Assumption

$\mathbf{m}_k$ is sparse

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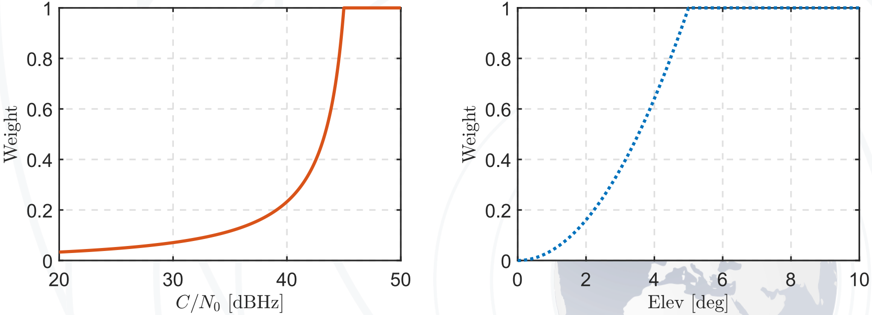
Weights for Navigation

Weights related to **signal strengths** and **satellite elevations**[12]



[12] Eugenio Realini and Mirko Reguzzoni. "goGPS: Open Source Software for Enhancing the Accuracy of Low-Cost Receivers by Single-Frequency Relative Kinematic Positioning". In: *Measurement Science and Technology* 24.11 (2013).

Additional Solutions

Avoid flickering in the estimation by **temporal smoothing**[13]

- Total variation (Fused LASSO)[14]

$$\arg\min_{\boldsymbol{\theta}_k} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_1 + \mu\|\boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k-1}\|_1$$

Robust estimation for the noise covariance matrix[15]

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[16] H. Kuusniemi, A. Wieser, G. Lachapelle, and J. Takala. "User-Level Reliability Monitoring in Urban Personal Satellite-Navigation". In: *IEEE Trans. Aerosp. Electron. Syst.* 43.4 (2007), pp. 1305–1318.

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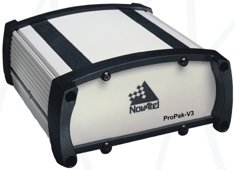
Proposed Strategies

Name	MP bias	Noise covariance
EKF	$\boldsymbol{m}_k = \boldsymbol{0}$	$\begin{bmatrix} \sigma_p^2 I_{s_k} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_r^2 I_{s_k} \end{bmatrix}$
Weighted LASSO	Weighted-ℓ_1	$\begin{bmatrix} \sigma_p^2 I_{s_k} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_r^2 I_{s_k} \end{bmatrix}$
Fused LASSO	Weighted-ℓ_1 and smoothing	$\begin{bmatrix} \sigma_p^2 I_{s_k} & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_r^2 I_{s_k} \end{bmatrix}$
Danish	$\boldsymbol{m}_k = \boldsymbol{0}$	Danish method
Weighted LASSO +Danish	Weighted-ℓ_1	Danish method
Fused LASSO +Danish	Weighted-ℓ_1 and smoothing	Danish method

Experimental setup

- Ground truth: Novatel SPAN (GPS receiver Propak-V3 + inertial measurements unit IMAR)

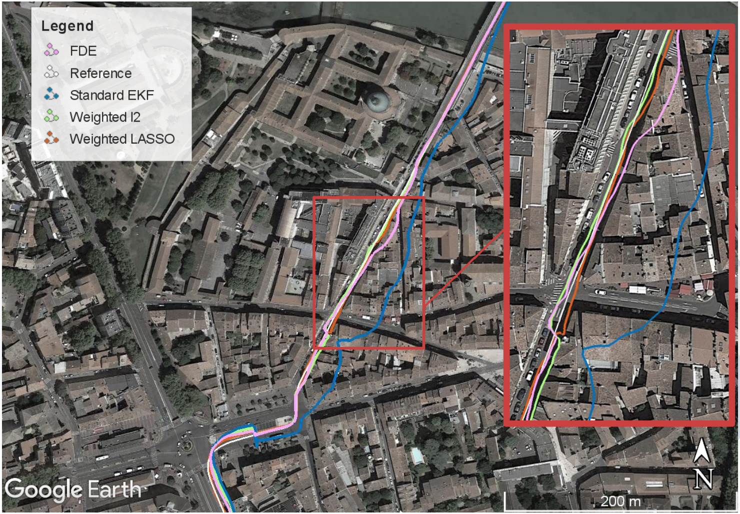


- Measurements: Ublox AEK-4T

Trajectory

Local Results

Few MP



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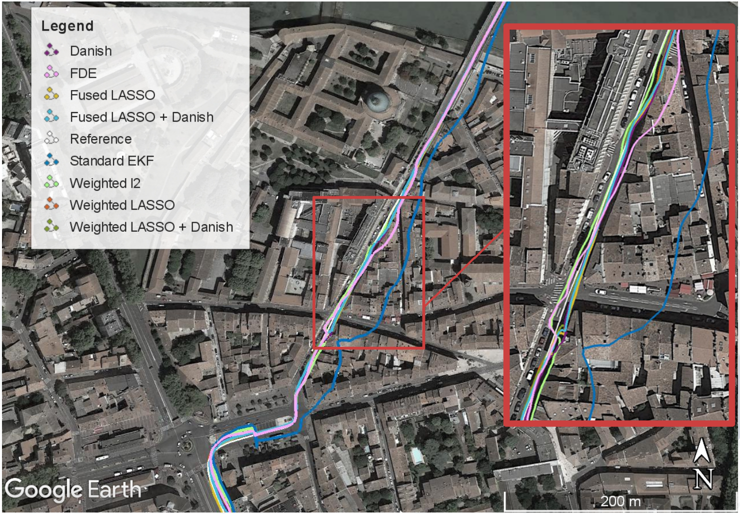
Few MP



Local Results

More MP

Local Results

More MP

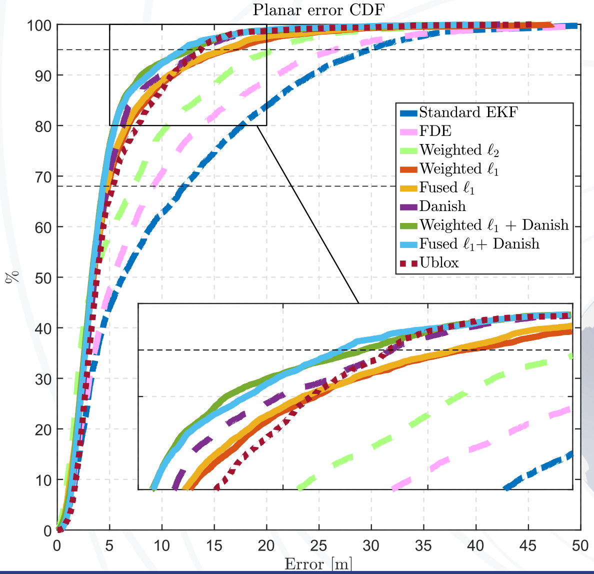
Local Results

More robust methods

Local Results

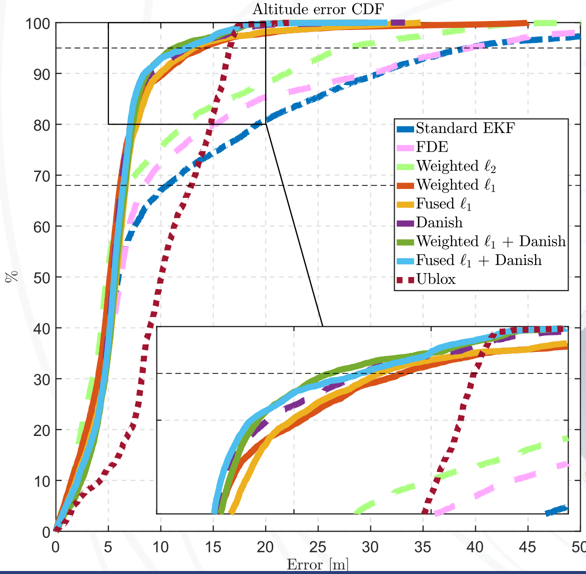
More robust methods

Global Results: Planar Error



Method	95-th %	Comparison
EKF	29.75m	148%
FDE	26.52m	121%
Weighted ℓ_2	20.67m	72%
Weighted ℓ_1	16.25m	35%
Fused ℓ_1	15.9 m	32%
Danish	13.75m	15%
Danish + Weighted ℓ_1	12.77m	6%
Danish + Fused ℓ_1	12.00m	0%
Ublox	13.67m	14%

Global Results: Altitude Error



Method	95-th %	Comparison
EKF	39.34m	218%
FDE	39.38m	218%
Weighted ℓ_2	27.63m	123%
Weighted ℓ_1	13.95m	13%
Fused ℓ_1	13.14 m	6%
Danish	13.75m	1%
Danish + Weighted ℓ_1	11.34m	-8%
Danish + Fused ℓ_1	12.38m	0%
Ublox	16.55m	34%

Tuning the hyperparameter

$$\underset{\boldsymbol{x}_k, \boldsymbol{m}_k}{\arg\min} \left\{ \frac{1}{2} \| \boldsymbol{y}_k - \boldsymbol{H}_k \boldsymbol{x}_k - \boldsymbol{m}_k \|_2^2 + \lambda_k \| \boldsymbol{W}_k \boldsymbol{m}_k \|_1 \right\}$$

Cross-validation

Tuning the hyperparameter

$$\underset{\boldsymbol{x}_k, \boldsymbol{m}_k}{\arg\min} \left\{ \frac{1}{2} \|\boldsymbol{y}_k - \boldsymbol{H}_k \boldsymbol{x}_k - \boldsymbol{m}_k\|_2^2 + \lambda_k \|\boldsymbol{W}_k \boldsymbol{m}_k\|_1 \right\}$$

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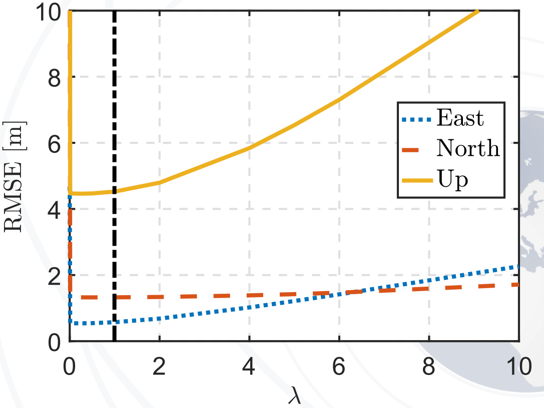

Bayesian Estimation

Bayesian Framework

Rewriting the problem

$$\underset{\boldsymbol{x}_k,\boldsymbol{m}_k}{\arg\min} \frac{1}{2}\|\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{x}_k - \boldsymbol{m}_k\|_2^2 + \lambda_k\|\boldsymbol{W}_k\boldsymbol{m}_k\|_1$$

$$\Leftrightarrow \underset{\boldsymbol{x}_k,\boldsymbol{m}_k}{\arg\max} \underbrace{\exp\left(-\frac{1}{2}\|\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{x}_k - \boldsymbol{m}_k\|_2^2\right)}_{\propto p(\boldsymbol{y}_k|\boldsymbol{x}_k,\boldsymbol{m}_k)} \underbrace{\exp\left(-\lambda_k\|\boldsymbol{W}_k\boldsymbol{m}_k\|_1\right)}_{\propto p(\boldsymbol{m}_k)}$$

- Gaussian likelihood $\boldsymbol{y}_k|\boldsymbol{x}_k, \boldsymbol{m}_k$
- Laplacian prior for $\boldsymbol{m}_k$

Missing

- Prior for $\boldsymbol{x}_k$ (assuming independence between $\boldsymbol{m}_k$ and $\boldsymbol{m}_k$)
- Hyperprior for λ_k

Bayesian Framework

Rewriting the problem

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Hierarchical Bayesian Model

Gaussian likelihood for $\boldsymbol{y}_k$ (from model)

$$\boldsymbol{y}_k | \boldsymbol{x}_k, \boldsymbol{m}_k \sim \mathcal{N}(\boldsymbol{y}_k; \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{m}_k, \boldsymbol{R}_k)$$

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$$m_{i,k} \sim \mathcal{L}\left(m_{i,k}; 0, \frac{1}{\lambda_k w_{i,k}}\right)$$

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Bayesian LASSO[17]

Introduction of latent variable $\boldsymbol{\tau}_k^2$

Posterior distribution

$$f(\boldsymbol{x}_k, \boldsymbol{m}_k, \boldsymbol{\tau}_k^2, \lambda_k^2 | \boldsymbol{y}_k) \propto \underbrace{f(\boldsymbol{y}_k | \boldsymbol{x}_k, \boldsymbol{m}_k)}_{\text{likelihood}} \underbrace{f(\boldsymbol{x}_k) f(\boldsymbol{m}_k | \boldsymbol{\tau}_k^2, \lambda_k^2) f(\boldsymbol{\tau}_k^2 | \lambda_k^2)}_{\text{priors}} \underbrace{f(\lambda_k^2)}_{\text{hyperprior}}$$

MAP estimator: Mode of this distribution

MMSE estimator: Mean of this distribution

$\rightarrow$ intractable

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[17] Trevor Park and George Casella. "The Bayesian Lasso". In: *Journal of the American Statistical Association* 103 (2008), pp. 681–686.

MCMC Methods

Markov Chain Monte Carlo[18] methods draw samples $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots$ from posterior distribution of $\boldsymbol{\theta}$

- posterior distribution is known up to a multiplicative constant
- samples $\boldsymbol{y}^{(t)}$ can be drawn from a proposal distribution
- set $\boldsymbol{\theta}^{(t)} = \boldsymbol{y}^{(t)}$ with an appropriate acceptance probability

Gibbs sampling

- proposal distributions are the conditional distributions
- acceptance probability is 1

[18] Christin Robert and George Casella. *Monte Carlo Statistical Methods*. Springer-Verlag New York, 2004.

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Conditional distributions

Latent variable $\tau_{i,k}^2 | m_{i,k}, \lambda_k^2 \sim \mathcal{GIG}(\tau_{i,k}^2; \frac{1}{2}, w_{i,k}^2 \lambda_k^2, m_{i,k}^2)$

Multipath bias $\boldsymbol{m}_k | \boldsymbol{y}_k, \boldsymbol{x}_k, \boldsymbol{\tau}_k^2 \sim \mathcal{N}(\boldsymbol{m}_k; \boldsymbol{\mu}_{\boldsymbol{m}_k}, \boldsymbol{\Sigma}_{\boldsymbol{m}_k})$

State vector variation $\boldsymbol{x}_k | \boldsymbol{y}_k, \boldsymbol{m}_k \sim \mathcal{N}(\boldsymbol{x}_k; \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{m}_k), \boldsymbol{P}_{k|k})$

Hyperparameter $\lambda_k^2 | \boldsymbol{\tau}_k^2 \sim \mathcal{G}\left(\lambda_k^2; 2s_k, \frac{1}{2} \sum_{i=1}^{2s_k} w_{i,k}^2 \tau_{i,k}^2\right)$

$$\boldsymbol{\Sigma}_{\boldsymbol{m}_k} = \operatorname{diag}\left(\frac{\sigma_{i,k}^2 \tau_{i,k}^2}{\sigma_{i,k}^2 + \tau_{i,k}^2}\right), \quad \boldsymbol{\mu}_{\boldsymbol{m}_k} = \operatorname{diag}\left(\frac{\tau_{i,k}^2}{\sigma_{i,k}^2 + \tau_{i,k}^2}\right)(\boldsymbol{y}_k - \boldsymbol{H}_k \boldsymbol{x}_k)$$

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T (\boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T + \boldsymbol{R}_k), \quad \boldsymbol{P}_{k|k} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k) \boldsymbol{P}_{k|k-1}$$

Multipath Detection/Estimation: Synthetic Data

Simulation scenario

- 200 Monte Carlo iterations
- States and measurements generated by system equations
- Artificial (controlled) dynamic MP biases

Gibbs sampler

- 1000 iterations with a 100 burn-in period
- Convergence assessment[19]: PSRF$<$1.2
- MMSE estimators: averages of generated samples

[19] Stephen P. Brooks and Andrew Gelman. "General Methods for Monitoring Convergence of Iterative Simulations". In: *Journal of Computational and Graphical Statistics* 7.4 (1998), pp. 434 –455.

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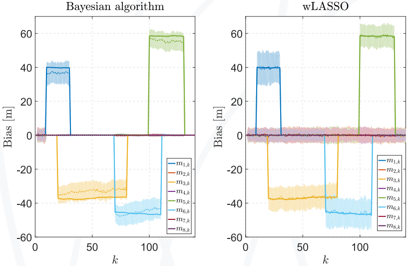
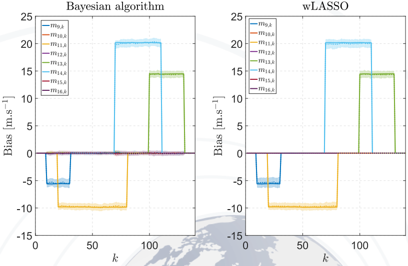
Multipath Detection/Estimation: Synthetic Data

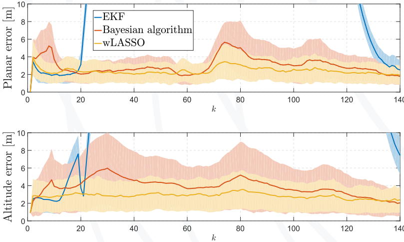
Pseudoranges



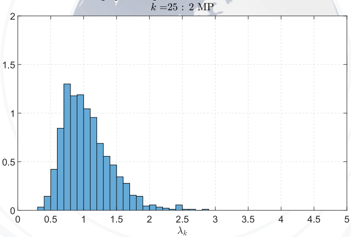
Pseudorange rates



Position



Hyperparameter



Multipath Detection/Estimation: Synthetic Data

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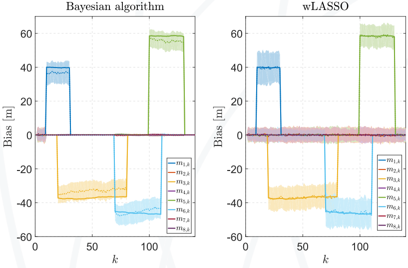

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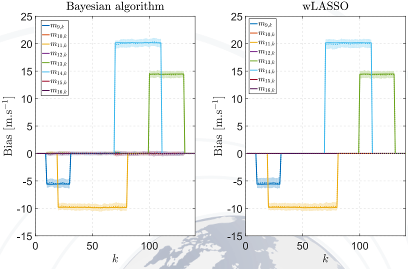

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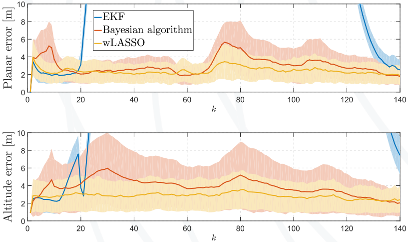
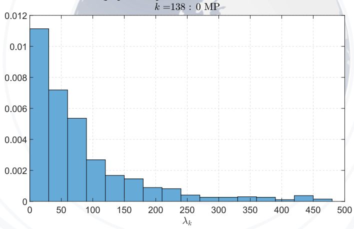
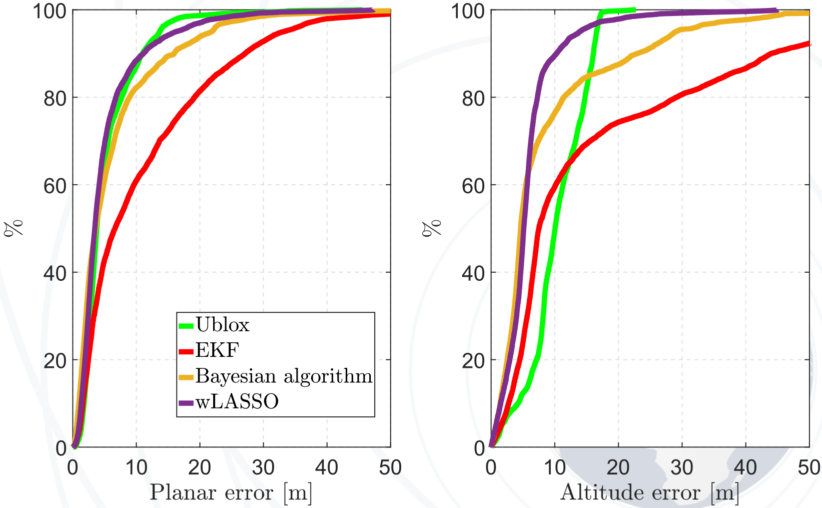


Hyperparameter



Multipath Detection/Estimation: Real Data



Mixture Models

Main idea

Generalized problem

$$\boldsymbol{z}_k = \boldsymbol{h_k}(\xi_k) + \underbrace{\boldsymbol{m}_k + \boldsymbol{n}_k}_{\boldsymbol{\nu}_k} \quad \Rightarrow \quad \begin{array}{c} \boldsymbol{m}_k \sim \mathcal{L}, \quad \boldsymbol{n}_k \sim \mathcal{N} \\ \boldsymbol{\nu}_k \sim \mathcal{D} \end{array}$$

Many distributions have been proposed

- Conditional Gaussian[20]
- Gaussian mixtures[21]
- Dirichlet process mixtures[22]

---

[20] S. Tay and J. Marais. "Weighting models for GPS Pseudorange observations for land transportation in urban canyons". In: *Proc. of the 6th European Workshop on GNSS Signals and Signal Processing*. Munich, Germany, 2013.

[21] N. Viandier, D. F. Nahimana, J. Marais, and E. Duflos. "GNSS Performance Enhancement in Urban Environment Based on Pseudo-range Error Model". In: *Proc. Symp. of the IEEE/ION Position, Location and Navigation*. Monterey, CA, 2008, pp. 377–382.

[22] A. Rabaoui, N. Viandier, E. Duflos, J. Marais, and P. Vanheeghe. "Dirichlet Process Mixtures for Density Estimation in Dynamic Nonlinear Modeling: Application to GPS Positioning in Urban Canyons". In: *IEEE Trans. Signal Process.* 60.4 (2012), pp. 1638–1655.

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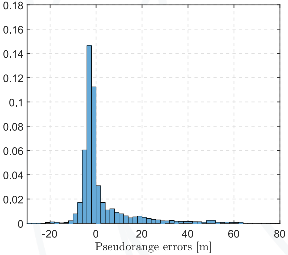
Gaussian Mixtures

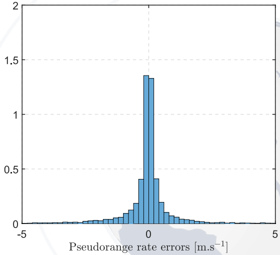
Model

$$n_{i,k} \sim \sum_{\ell=1}^{M} \alpha_{i,\ell} \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \Leftrightarrow \begin{cases} P(c_{i,k} = \ell) = \alpha_{i,\ell} \\ n_{i,k} | c_{i,k} = \ell \sim \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \end{cases}$$

Estimation

- Expectation Maximization method[23]





[23] A. P. Dempster, N. M. Laird, and D. B. Rubin. "Maximum Likelihood from Incomplete Data via the EM Algorithm". In: *Journal of the Royal Statistical Society, Series B (Methodological)* 39.1 (1977), pp. 1–38.

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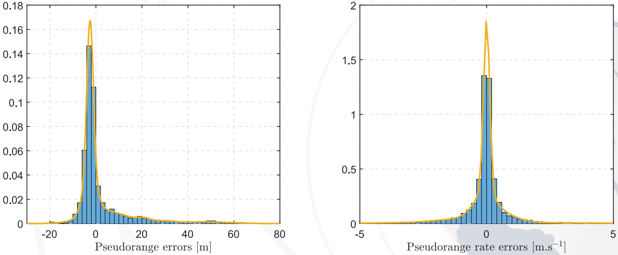


[23] A. P. Dempster, N. M. Laird, and D. B. Rubin. "Maximum Likelihood from Incomplete Data via the EM Algorithm". In: *Journal of the Royal Statistical Society, Series B (Methodological)* 39.1 (1977), pp. 1–38.

Hidden Markov Model

Principle

- Gaussian mixtures with dependance on previous state $k-1$

Model

$$n_{i,k} \sim \sum_{j=1}^{M} \alpha_{i,j} \mathcal{N}(n_{i,k}; \mu_{i,j}, \sigma_{i,j}^2) \Leftrightarrow \left\{ \begin{array}{c} P(c_{i,k} = \ell | c_{i,k-1} = m) = (\boldsymbol{A}_i)_{m,\ell} \\ P(c_{i,0} = \ell) = (\boldsymbol{\Pi}_i)_\ell \\ n_{i,k} | c_{i,k} = \ell \sim \mathcal{N}(n_{i,k}; \mu_{i,\ell}, \sigma_{i,\ell}^2) \end{array} \right.$$

Estimation

- Baum-Welch method[24]

[24] Lawrence R. Rabiner. "A Tutorial on Hidden Markov Models and Selected Applications in Speech recognition". In: *Proceedings of the IEEE* 77.2 (1989), pp. 257–286.

Filters

Gaussian Mixtures

- Gaussian sum filter[25]: bank of Kalman filters for all modes of the mixtures

HMM

- Interacting Multiple Model[26]: bank of Kalman filters for all modes of the mixtures using approximations

Computing limitations

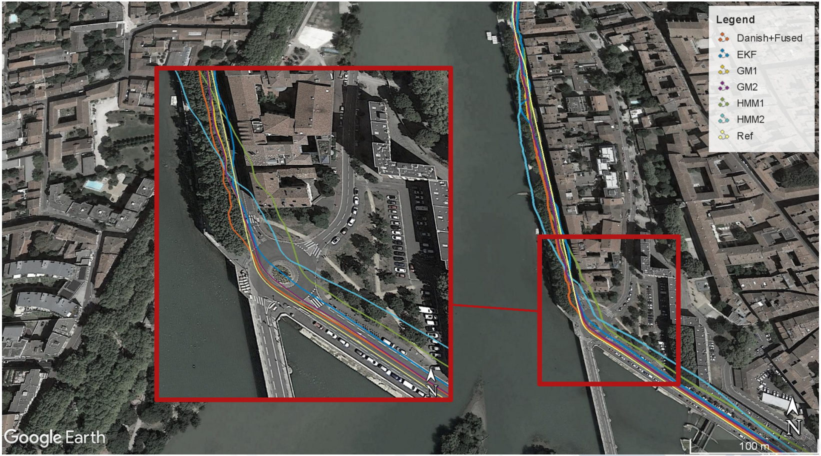
- Huge number of modes: M^{2s_k}
- Limitation to a maximum of two mode changes

[25] Daniel L. Alspach and Harold W. Sorenson. "Nonlinear Bayesian Estimation Using Gaussian Sum Approximations". In: *IEEE Trans. Autom. Contr.* 17.4 (1972), pp. 439–448.

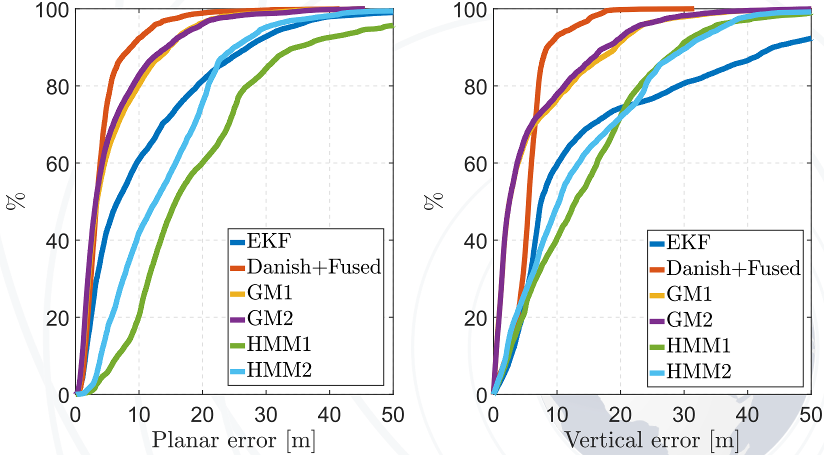
[26] Yaakov Bar-Shalom, Subhash Challa, and Henk A. P. Blom. "IMM Estimator Versus Optimal Estimator for Hybrid Systems". In: *IEEE Trans. Aerosp. Electron. Syst.* 41.3 (2005), pp. 986–991.

Some Experiments

Some Experiments



Cumulative Distribution Functions



Conclusions and Future Works

Sparse Estimation

Advantages

- Joint detection/estimation of MP bias
- Only need raw measurements (RINEX) from any receiver
- Real-time formulation
- Can be combined to robust estimation

Drawback

- Hyperparameter tuning

Future work

- Other weighting matrices
- Other hyperparameter estimation: time-dependent, DOP-dependent, ...
- Fusion with other sensors/signals: multi constellation, multi frequency, vision, 5G, ...

Bayesian Estimation

Advantages

- No hyperparameter tuning
- Measures of uncertainties

Drawback

- Computationally intensive

Future work

- Assign different priors to multipath biases
- More informative priors for the hyperparameter
- Develop more efficient algorithms: SMC methods

Mixture Models

Advantages

- More flexibility
- Straightforward computations in the Gaussian case

Drawbacks

- Full solution computationally intensive: reduce the number of births and deaths
- Prior learning of the noise distribution

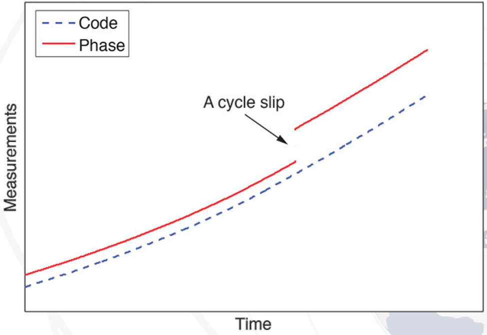
Future work

- Online estimation of the mixtures
- Optimize the mode configurations: MCMC, particular filters
- Combine sparse estimation and Gaussian mixtures

Sparsity in GNSS

Precise Point Positioning in urban environment

- Multi-frequency signals: instantaneous ambiguity resolution[27]
- Use of sparse estimation to detect cycle slips

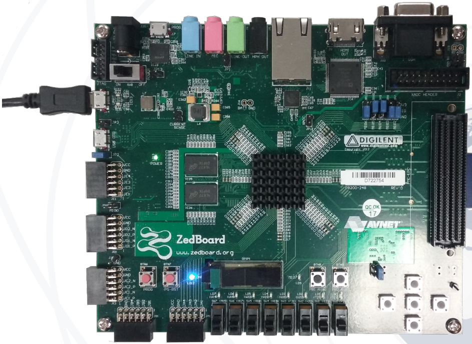


[27] D. Laurichesse and S. Banville. "Innovation: Instantaneous Centimeter-Level Multi-Frequency Precise Point Positioning". In: *GPS World* (2018).

Sparsity in GNSS

Software Define Radio

- Versatile device
- Implement sparse estimation earlier in the receiver

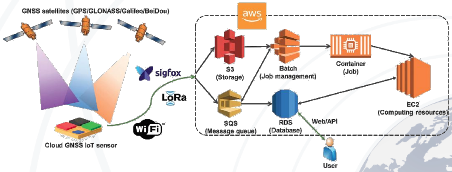


ISAE Supaero

Sparsity in GNSS

Collaborative Positioning

- Increasing number of IoT sensors
- Stock and share data: cloud[28]



Integrity

- Develop integrity criteria based on sparse estimation
- Spoofing and jamming detection/correction
- Authentication of the signals

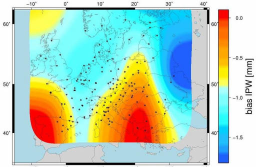
[28] V. Lucas-Sabola, G. Seco-Granados, J. A. López-Salcedo, and J. A. Garciá-Molina. "GNSS IoT Positioning: From Conventional Sensors to a Cloud-Based Solution". In: *Inside GNSS* (2018).

Thanks for your attention!

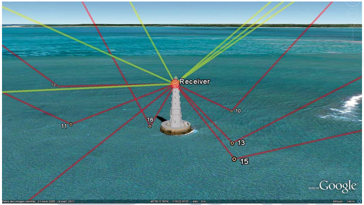
Back-up

- Increasingly various GPS applications
- GPS Signal
- Extended Kalman Filter
- Solving the Sparse Bias Problem
- Discontinuities in Estimation
- The ℓ_0 Problem
- Comparison with reweighted-ℓ_1
- Wavelet decomposition
- Bayesian LASSO
- Hierarchical Bayesian Model with MP indicator
- Multipath Detection/Estimation: Hyperparameter evolution
- Gaussian Mixtures

Increasingly various GPS applications

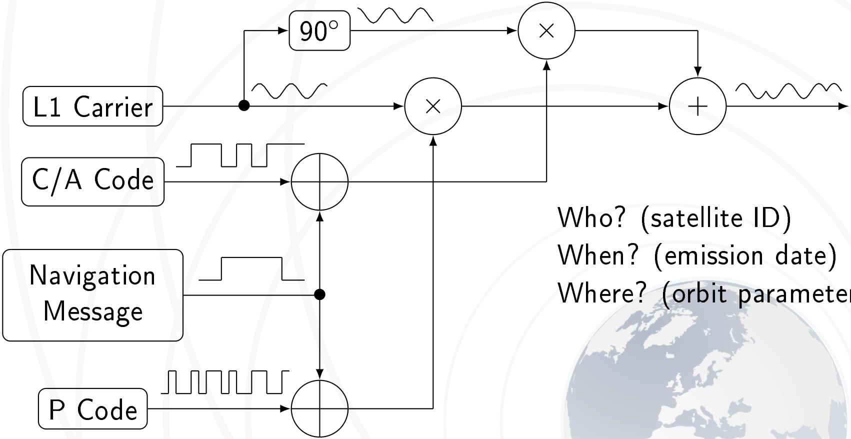


A. Brzezinski et al., "Geodetic and Geodynamic Studies at Department of Geodesy and Geodetic Astronomy Wut", in *Reports on Geodesy and Geoinformatics* vol. 100, March 2016, pp.165-200



Marielle Mayo, "GNSS-R Signaux réfléchis", in *Géomètre* n°2123, March 2015, pp.46-49

GPS Signal



Who? (satellite ID)
When? (emission date)
Where? (orbit parameters)

Extended Kalman Filter

State propagation $\xi_k \in \mathbb{R}^8$

Hypothesis: random walk

$$\xi_k = \boldsymbol{F}_k \xi_{k-1} + \boldsymbol{u}_k \quad \text{with} \quad \begin{array}{l} \boldsymbol{F}_k \text{ known} \\ \boldsymbol{u}_k \sim \mathcal{N}(\boldsymbol{n}_k; 0, \boldsymbol{Q}_k) \end{array}$$

EKF= Kalman Filter + Linearization

Kalman predictions

$$\hat{\xi}_{k|k-1} = \boldsymbol{F}_k \hat{\xi}_{k-1|k-1} \qquad \rightarrow \text{Linearization point}$$

$$\boldsymbol{P}_{k|k-1} = \boldsymbol{F}_k \boldsymbol{P}_{k-1|k-1} \boldsymbol{F}_k + \boldsymbol{Q}_k$$

Kalman updates

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T (\boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T + \boldsymbol{R}_k)$$

$$\hat{\xi}_k = \hat{\xi}_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{z}_k - \boldsymbol{h}_k(\hat{\xi}_{k|k-1}) - \boldsymbol{m}_k)$$

$$\boldsymbol{P}_{k|k} = (I - \boldsymbol{K}_k \boldsymbol{H}_k) \boldsymbol{P}_{k|k-1}$$

Tuning the EKF

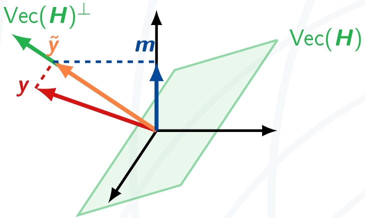
State Evolution

$$\boldsymbol{F}_{k-1} = \begin{bmatrix} \boldsymbol{I}_4 & \Delta t_k \\ 0 & \boldsymbol{I}_4 \end{bmatrix}$$

State Covariance

$$\boldsymbol{Q}_{k-1} = \begin{bmatrix} S_a \frac{\Delta t_{k-1}^3}{3} I_3 & \mathbf{0}_{3\times1} & S_a \frac{\Delta t_{k-1}^2}{2} I_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & c^2 \left(S_b \Delta t_{k-1} + S_d \frac{\Delta t_{k-1}^3}{3} \right) & \mathbf{0}_{1\times3} & c^2 \left(S_d \frac{\Delta t_{k-1}^2}{2} \right) \\ S_a \frac{\Delta t_{k-1}^2}{2} I_3 & \mathbf{0}_{3\times1} & S_a \Delta t_{k-1} I_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & c^2 \left(S_d \frac{\Delta t_{k-1}^2}{2} \right) & \mathbf{0}_{1\times3} & c^2 (S_d \Delta t_{k-1}) \end{bmatrix}$$

Solving the Sparse Bias Problem



Measurements:

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{m}_k + \boldsymbol{n}_k$$

Profile likelihood:

$$\boldsymbol{x}_{\boldsymbol{k}} = (\boldsymbol{H}_k^T \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T (\boldsymbol{y}_k - \boldsymbol{m}_k)$$

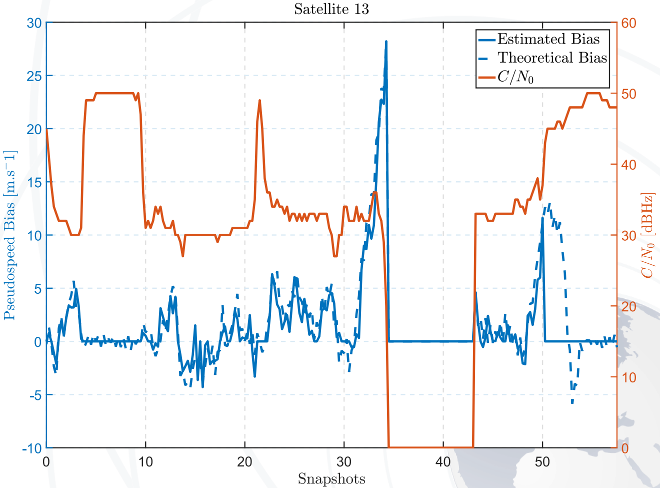
$$\arg\min_{\boldsymbol{x}_k, \boldsymbol{m}_k} \frac{1}{2} \| \boldsymbol{y}_k - \boldsymbol{H}_k \boldsymbol{x}_k - \boldsymbol{m}_k \|_2^2 + \lambda_k \| \boldsymbol{W}_k \boldsymbol{m}_k \|_1$$

$$\arg\min_{\boldsymbol{m}_k} \frac{1}{2} \| \boldsymbol{y}_k - \underbrace{\boldsymbol{H}_k (\boldsymbol{H}_k^T \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T}_{\boldsymbol{P}_k} (\boldsymbol{y}_k - \boldsymbol{m}_k) - \boldsymbol{m}_k \|_2^2 + \lambda_k \| \boldsymbol{W}_k \boldsymbol{m}_k \|_1$$

$$\arg\min_{\boldsymbol{m}_k} \frac{1}{2} \| \underbrace{(\boldsymbol{I} - \boldsymbol{P}_k) \boldsymbol{y}_k}_{\tilde{\boldsymbol{y}}_k} - \underbrace{(\boldsymbol{I} - \boldsymbol{P}_k) \boldsymbol{W}_{\boldsymbol{k}}^{-1}}_{\tilde{\boldsymbol{H}}_k} \underbrace{\boldsymbol{W}_{\boldsymbol{k}} \boldsymbol{m}_k}_{\boldsymbol{\theta}_k} \|_2^2 + \lambda_k \| \underbrace{\boldsymbol{W}_k \boldsymbol{m}_k}_{\boldsymbol{\theta}_k} \|_1$$

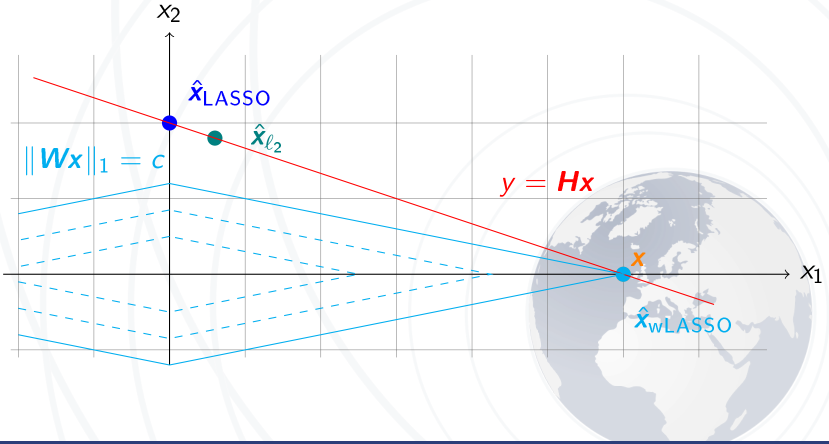
$$\arg\min_{\boldsymbol{\theta}_k} \frac{1}{2} \| \tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{\theta}_k \|_2^2 + \lambda_k \| \boldsymbol{\theta}_k \|_1 \rightarrow \text{LASSO problem}$$

Discontinuities in Estimation

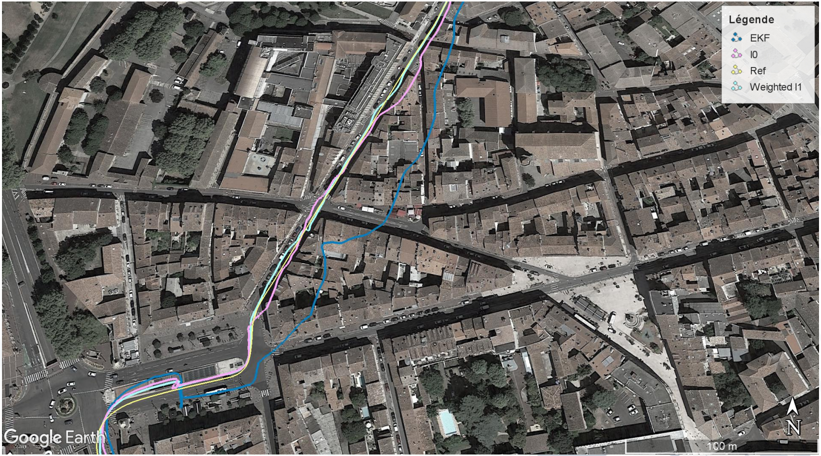


Non-convex Problem

$$\arg\min_{\boldsymbol{\theta}_k} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k\boldsymbol{\theta}_k\|_2^2 + \lambda_k\|\boldsymbol{\theta}_k\|_0$$



Results



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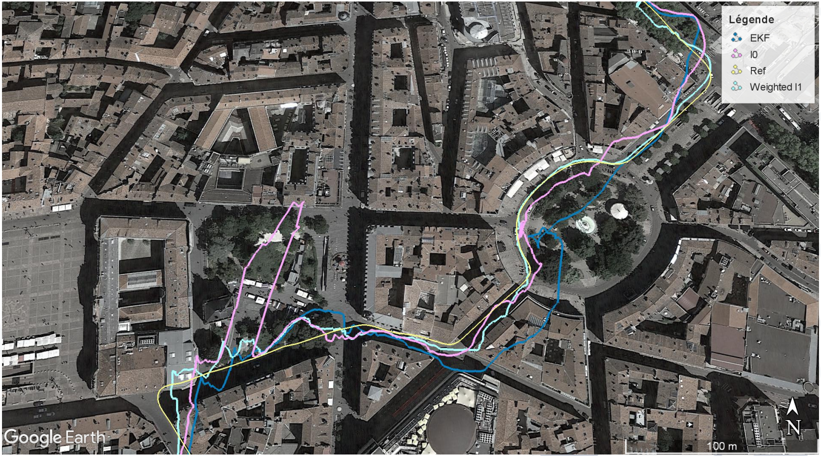

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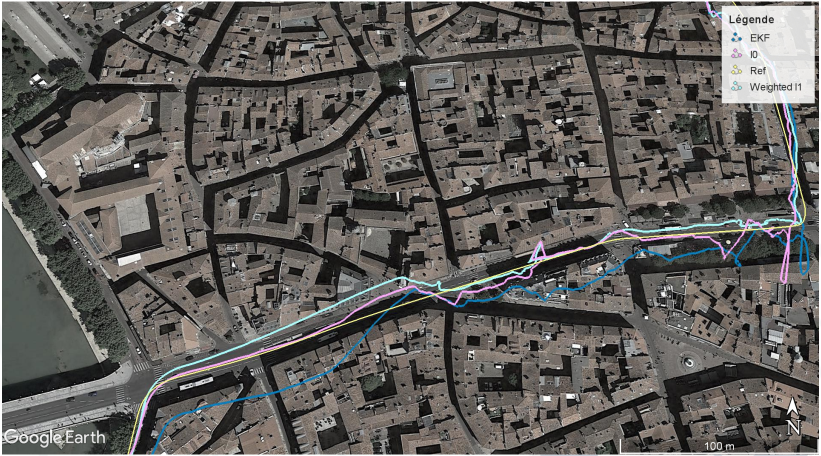


Reweighted-ℓ_1

Initialize $\boldsymbol{W}$

for $\ell = 0, \dots, \ell_{\max}$ **do**

Solve $\boldsymbol{\theta}^{(\ell)} = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{R}^n} \frac{1}{2} \|\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{H}}\boldsymbol{\theta}\|_2^2 + \lambda \|\boldsymbol{W}^{(\ell)}\boldsymbol{\theta}\|_1$

Update weights

for $i = 1, \dots, n$ **do**

$w_i^{(\ell+1)} = \frac{1}{\theta_i^{(\ell)} + \epsilon}$

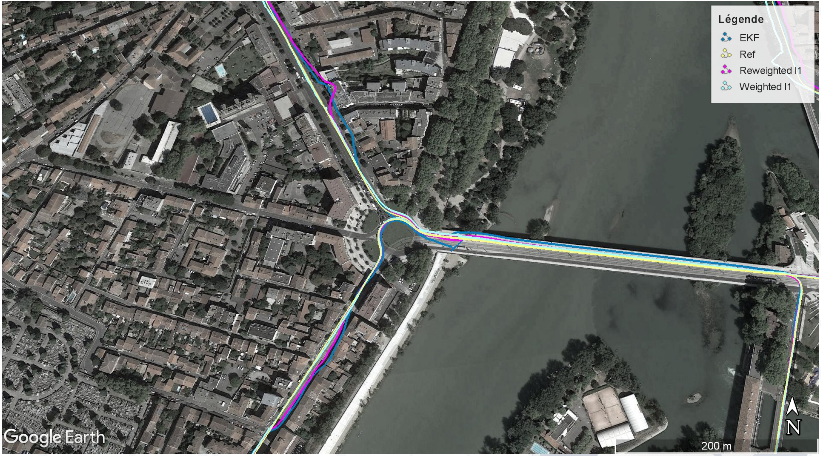
end for

end for

Results


Légende
EKF
Ref
Reweighted l1
Weighted l1
Google Earth
200 m
N

Results


Légende
EKF
Ref
Reweighted l1
Weighted l1
Google Earth
200 m
N

Results


Légende
EKF
Ref
Reweighted l1
Weighted l1
Google Earth
200 m
N

Wavelets decomposition

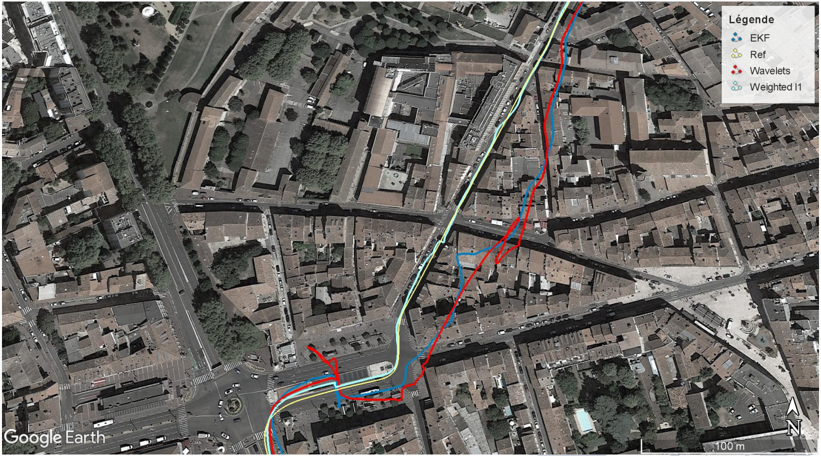
$$\arg\min_{\boldsymbol{m}_k} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{m}_k\|_2^2 + \lambda_k \|\boldsymbol{W}_k \boldsymbol{m}_k\|_1$$

$$\arg\min_{\boldsymbol{m}_k} \frac{1}{2}\|\tilde{\boldsymbol{y}}_k - \tilde{\boldsymbol{H}}_k \boldsymbol{m}_k\|_2^2 + \lambda_k \|\psi_k \boldsymbol{m}_k\|_1 \qquad (1)$$

$$(2)$$

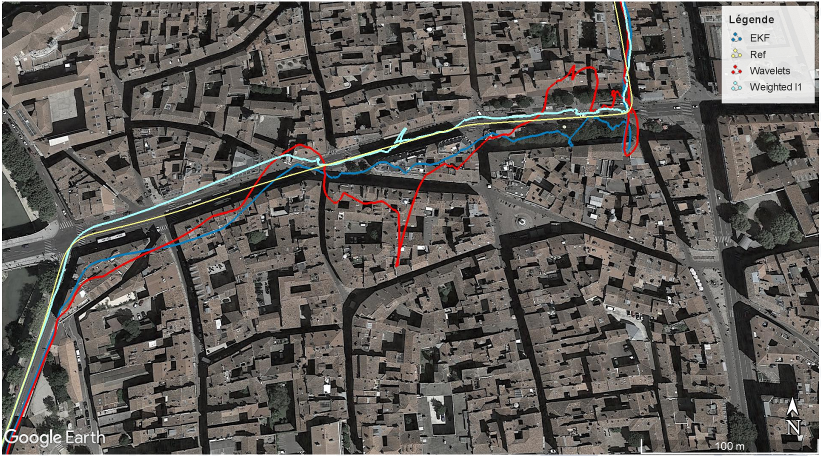
Collaboration with Universidad Industrial de Santander (Columbia)

Results



Results

Results



Bayesian LASSO

Completion (marginalization trick)

$$\frac{w_{i,k}\lambda_k}{2}\exp\left(-w_{i,k}\lambda_k|m_{i,k}|\right) = \int_0^{+\infty}\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{m_{i,k}^2}{2s}\right)\frac{w_{i,k}^2\lambda_k^2}{2}\exp\left(-\frac{w_{i,k}^2\lambda_k^2 s}{2}\right)ds$$

$$m_{i,k}|\lambda_k^2 \sim \mathcal{L}\left(m_{i,k};0,\frac{1}{\lambda_k w_{i,k}}\right) \Leftrightarrow \exists\tau_{i,k}^2, \begin{cases} m_k|\tau_{i,k}^2 \sim \mathcal{N}(m_{i,k};0,\tau_{i,k}^2) \\ \tau_{i,k}^2|\lambda_k^2 \sim \mathcal{E}\left(\tau_{i,k}^2;\frac{2}{\lambda_k^2 w_{i,k}^2}\right)\end{cases}$$

Posterior distribution

$$f(\boldsymbol{x}_k,\boldsymbol{m}_k,\boldsymbol{\tau}_k^2,\lambda_k^2|\boldsymbol{y}_k) \propto \underbrace{f(\boldsymbol{y}_k|\boldsymbol{x}_k,\boldsymbol{m}_k)}_{\text{likelihood}}\underbrace{f(\boldsymbol{x}_k)f(\boldsymbol{m}_k|\boldsymbol{\tau}_k^2,\lambda_k^2)f(\boldsymbol{\tau}_k^2|\lambda_k^2)}_{\text{priors}}\underbrace{f(\lambda_k^2)}_{\text{hyperprior}}$$

Hierarchical Bayesian Model with MP indicator

$$\boldsymbol{y}_k | \boldsymbol{x}_k, \boldsymbol{m}_k, \sim \mathcal{N}(\boldsymbol{y}_k; \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{m}_k, \boldsymbol{R}_k)$$

$$\boldsymbol{x}_k \sim \mathcal{N}(\boldsymbol{x}_k; \boldsymbol{0}, \boldsymbol{P}_{k|k-1})$$

$$m_{i,k} | b_{i,k}, \tau_{i,k}^2 \sim \begin{cases} \delta(m_{i,k}) & \text{if } b_{i,k} = 0 \\ \mathcal{N}\left(m_{i,k}; 0, \tau_{i,k}^2\right) & \text{if } b_{i,k} = 1 \end{cases}, i = 1, \ldots, 2s_k$$

$$\tau_{i,k}^2 | \lambda_k^2 \sim \mathcal{E}\left(\tau_{i,k}^2; \frac{2}{\lambda_k^2 w_{i,k}^2}\right), i = 1, \ldots, 2s_k$$

$$b_{i,k} | p_k \sim \mathcal{B}(b_{i,k}; p_k), i = 1, \ldots, 2s_k$$

$$p_k \sim \mathcal{U}_{[0,1]}(p_k)$$

$$f(\lambda_k^2) \propto \frac{1}{\lambda_k^2}$$

Conditional distributions with MP indicator

Latent variable

$$\tau_{i,k}^2 | m_{i,k}, \lambda_k^2, b_{i,k} \begin{cases} \mathcal{E}\left(\tau_{i,k}^2; \frac{2}{w_{i,k}^2 \lambda_k^2}\right) & \text{if } b_{i,k} = 0 \\ \mathcal{GIG}\left(\tau_{i,k}^2; \frac{1}{2}, w_{i,k}^2 \lambda_k^2, m_{i,k}^2\right) & \text{if } b_{i,k} = 1 \end{cases}$$

Multipath indicator

$$b_{i,k} | y_{i,k}, \boldsymbol{x}_k, \tau_{i,k}^2, p_k = \mathcal{B}\left(b_{i,k} \,\middle|\, \frac{v_{i,k}}{u_{i,k} + v_{i,k}}\right)$$

Multipath bias

$$\boldsymbol{m}_k | \boldsymbol{y}_k, \boldsymbol{x}_k, \boldsymbol{\tau}_k^2 \sim \begin{cases} \delta(m_{i,j}) & \text{if } b_{i,k} = 0 \\ \mathcal{N}(\mu_{m_{i,k}}, \sigma_{m_{i,k}}^2) & \text{si } b_{i,k} = 1 \end{cases}$$

State vector variation

$$\boldsymbol{x}_k | \boldsymbol{y}_k, \boldsymbol{m}_k \sim \mathcal{N}(\boldsymbol{x}_k; \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{m}_k), \boldsymbol{P}_{k|k})$$

Hyperparameter λ_k

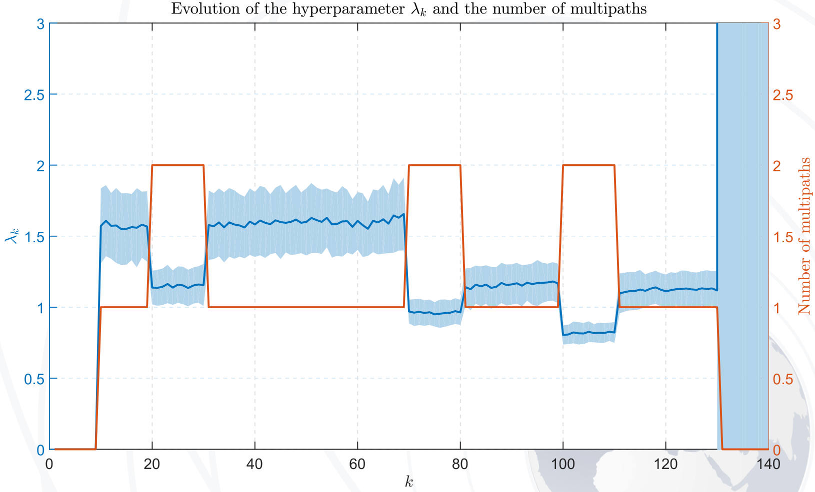
$$\lambda_k^2 | \boldsymbol{\tau}_k^2 \sim \mathcal{G}\left(\lambda_k^2; 2s_k, \frac{1}{2} \sum_{i=1}^{2s_k} w_{i,k}^2 \tau_{i,k}^2\right)$$

Hyperparameter p_k

$$f(p_k | \boldsymbol{b}_k) = \mathcal{B}e\left(p_k; \|\boldsymbol{b}_k\|_0 + 1, 2s_k - \|\boldsymbol{b}_k\|_0 + 1\right)$$

$$u_{i,k} = (1 - p_k), \quad v_{i,k} = p_k \sqrt{\frac{\sigma_{m_{i,k}}^2}{\tau_{i,k}^2}} \exp\left(\frac{\mu_{m_{i,k}}^2}{2\sigma_{m_{i,k}}^2}\right)$$

Multipath Detection/Estimation: Hyperparameter evolution



Gaussian Mixtures

Parameters : $\boldsymbol{A_i}, \boldsymbol{\Pi}_i, \boldsymbol{\mu}_i, \boldsymbol{\sigma}_i^2$

Used online	Need to learn the distributions
3 modes per satellite	M modes per measurement
Modes evolution estimated before (C/N_0 values)	Modes evolution estimated after (MAP estimator)
$\boldsymbol{A_i}$ and $\boldsymbol{\Pi}_i$ are the proportions $\boldsymbol{\mu}_i, \boldsymbol{\sigma}_i^2$ estimated via residuals	$\boldsymbol{A_i}, \boldsymbol{\Pi}_i, \boldsymbol{\mu}_i, \boldsymbol{\sigma}_i^2$ are estimated via Baum-Welch
Particle Filter	Bank of Kalman filters