Generate Random Samples from von Mises-Fisher and Watson Distributions

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1 Formula

The following derivation is based on the normal-tangent decomposition property of the distribution on sphere. Let $f(\mathbf{x};\boldsymbol{\mu})$ to be the p.d.f. of the distribution where μ is the mean direction. The random variable x can be decomposed as:

$$\begin{aligned}\mathbf{x} &= (\mathbf{x}^T\boldsymbol{\mu})\boldsymbol{\mu} + (I_p - \boldsymbol{\mu}\boldsymbol{\mu}^T)\mathbf{x} \\ &= (\mathbf{x}^T\boldsymbol{\mu})\boldsymbol{\mu} + \|I_p - \boldsymbol{\mu}\boldsymbol{\mu}^T\| S_{\boldsymbol{\mu}}(\mathbf{x}),\end{aligned} \tag{1}$$

where $S_{\boldsymbol{\mu}}(\mathbf{x}) = (I_p - \boldsymbol{\mu}\boldsymbol{\mu}^T)\mathbf{x}/\|(I_p - \boldsymbol{\mu}\boldsymbol{\mu}^T)\mathbf{x}\|$. Under any rotationally symmetric distribution, $S_{\boldsymbol{\mu}}(\mathbf{x})$ is uniformly distributed on $S^{p-2}_{\boldsymbol{\mu}^\perp}$ and the density of $t = \mathbf{x}^T\boldsymbol{\mu}$ is given by:

$$t \mapsto cf(t)(1-t^2)^{(p-3)/2} \tag{2}$$

Von Mises-Fisher Distribution

The p.d.f. of VMF distribution has the following form:

$$f(\mathbf{x};\boldsymbol{\mu},\kappa) = (\frac{\kappa}{2})^{p/2-1}\frac{1}{\Gamma(p/2)I_{p/2-1}(\kappa)}\exp\{\kappa\boldsymbol{\mu}^T\mathbf{x}\} \tag{3}$$

Let $\mathbf{x} = t\boldsymbol{\mu} + (1-t^2)^{1/2}\boldsymbol{\xi}$ and substitute in (3) we have:

$$f(t,\boldsymbol{\xi};\boldsymbol{\mu},\kappa) = (\frac{\kappa}{2})^{p/2-1}\frac{1}{\Gamma(p/2)I_{p/2-1}(\kappa)}\exp\{\kappa t\} \tag{4}$$

According to (2), we know that the density of t is proportional to $\exp\{\kappa t\}(1-t^2)^{(p-3)/2}$ and the integration from -1 to 1 should be equal to 1:

$$\begin{aligned} &\int_{-1}^{1} f(t)dt = C\int_{-1}^{1}\exp\{\kappa t\}(1-t^2)^{(p-3)/2}dt = 1 \\ \Rightarrow &C = \left(\int_{-1}^{1}\exp\{\kappa t\}(1-t^2)^{(p-3)/2}dt\right)^{-1} \end{aligned} \tag{5}$$

From the equation 9.6.18 of [1], we have

$$I_\nu(\kappa) = \frac{(\kappa/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1}\exp\{\kappa t\}(1-t^2)^{\nu-\frac{1}{2}}dt \tag{6}$$

Let $\nu = p/2-1$ in (6)

$$\begin{aligned} &\int_{-1}^{1}\exp\{\kappa t\}(1-t^2)^{(p-3)/2}dt = (\kappa/2)^{-(p/2-1)}I_{p/2-1}(\kappa)\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2}) \\ \Rightarrow &C = (\frac{\kappa}{2})^{(p/2-1)}\left(I_{p/2-1}(\kappa)\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})\right)^{-1} \\ \Rightarrow &f(t) = (\frac{\kappa}{2})^{(p/2-1)}\left(I_{p/2-1}(\kappa)\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})\right)^{-1}\exp\{\kappa t\}(1-t^2)^{(p-3)/2} \end{aligned} \tag{7}$$

Notice that the subscript $p/2-1$ of Bessel function I is different and should be corrected from (9.3.12) in Mardia's book [2], which is $(p-1)/2$.

Watson Distribution

The p.d.f. of Watson distribution has the following form:

$$f(\mathbf{x};\boldsymbol{\mu},\kappa) = \mathbb{M}(\frac{1}{2},\frac{p}{2},\kappa)^{-1}\exp\{\kappa(\boldsymbol{\mu}^T\mathbf{x})^2\} \tag{8}$$

Follow the similar flow, we substitute x by $t\boldsymbol{\mu} + (1-t^2)^{1/2}\boldsymbol{\xi}$ and obtain

$$f(t,\boldsymbol{\xi};\boldsymbol{\mu},\kappa) = \mathbb{M}(\frac{1}{2},\frac{p}{2},\kappa)^{-1}\exp\{\kappa t^2\} \tag{9}$$

The density of t is proportional to $\exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}$:

$$\begin{aligned} \int_{-1}^{1} f(t)dt = C\int_{-1}^{1} \exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}dt = 1 \\ \Rightarrow C = \left(\int_{-1}^{1} \exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}dt\right)^{-1} \end{aligned} \tag{10}$$

According to (13.2.1) in [1], we have

$$\begin{aligned} \frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}\mathbb{M}(a,b,\kappa) &= \int_{0}^{1} \exp\{\kappa x\}x^{a-1}(1-x)^{b-a-1}dx \\ &= \int_{0}^{1} exp\{\kappa t^2\}t^{2a-2}(1-t^2)^{b-a-1}2tdt \end{aligned} \tag{11}$$

Let $a = 1/2, b = p/2$ in (11)

$$\begin{aligned} \frac{\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{p}{2})}\mathbb{M}(\frac{1}{2},\frac{p}{2},\kappa) &= 2\int_{0}^{1} exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}dt \\ &= \int_{-1}^{1} exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}dt \end{aligned} \tag{12}$$

Substitute (12) into (10)

$$\begin{aligned} C &= \left(\int_{-1}^{1} \exp\{\kappa t^2\}(1-t^2)^{(p-3)/2}dt\right)^{-1} \\ &= \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})}\frac{1}{\mathbb{M}(\frac{1}{2},\frac{p}{2},\kappa)} \\ \Rightarrow f(t) &= \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{p-1}{2})\Gamma(\frac{1}{2})}\frac{1}{\mathbb{M}(\frac{1}{2},\frac{p}{2},\kappa)}\exp\{\kappa t^2\}(1-t^2)^{(p-3)/2} \end{aligned} \tag{13}$$

2 Algorithm

Given the target $\boldsymbol{\mu}$ and k for the von Mises-Fisher or Watson distribution, we can generate the random samples from the following steps:

1. Draw the vector $\mathbf{r}$ from the uniform distribution in $(p-1)$-dimension unit sphere and concatenate a 0 as the first element to $\mathbf{r}$ to get the p-dimension vector $\mathbf{u} = (0, \mathbf{r}) \in \mathcal{R}^p$.

2. Draw the scalar t from Eq.7 (VMF) or Eq.13. This step requires rejection sampling.

Since the density function has supoort as $[0, 1]$ and is bounded, rejection sampling is feasible.

3. Combine t and $\mathbf{u}$ according to Eq.2 to get the random variable $\mathbf{v}$.

4. Notice that $\mathbf{v}$ generated from the last step are from the VMF distribution with $\boldsymbol{\mu} = \mathbf{e}1$. We have to rotate the generated samples to the target mean direction.

References

[1] M. Abramowitz, I. A. Stegun, and others, *Handbook of mathematical functions.* Dover New York, 1972, vol. 1, no. 5.

[2] K. V. Mardia and P. E. Jupp, “Directional statistics,” 1999. [Online]. Available: http://cds.cern.ch/record/1254260